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OPTICAL TRANSMISSION ACROSS A ROUGH SURFACE

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Technical Report

by

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ABSTRACT

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I. INTRODUCTION

A comprehensive study of the sea requires investigations in both the microwave and optical portions of the spectrum. Thus, microwave radiometry provides a sensitive all-weather tool for the calculation of boundary temperatures and sea states, while subsurface properties can be probed by the spectral variation of sea color. For example, the chlorophyll content in the ocean,[1] the depth of coastal waters,[2] and the degree of pollution in coastal and inland waters all may be identifiable by their spectral signatures.

The problem of sea color is very complex. Sunlight and skylight are both scattered by the sea surface and transmitted across the boundary into the water medium. Some of this transmitted energy is backscattered, either by molecular and particulate matter in deep water or by the bottom in shallow waters, so that a certain fraction, the upwelling sea light, is transmitted back across the boundary into the air. Sea-color measurements must be made at angles outside the glitter pattern of the scattered sunlight, so the sea color is determined by the two components of scattered skylight and upwelling sea light.

The actual analysis of the sea-color problem must account for surface roughness. Thus, sunlight incident on a flat ocean surface will be transmitted in the single direction determined by Snell's law. For the general ocean boundary, however, Snell's law must be applied to each of the locally flat elements comprising the surface. This results in an angular distribution of transmitted energy, which, of course, will produce a final component of upwelling sea light with angular distribution dependent on the surface sea state. The scattered skylight will also have an angular radiation pattern peculiar to the sea state. Quite clearly, if subsurface properties are of primary interest, we want to make sea-color observations in directions where the scattered skylight does not dominate the upwelling sea light. Thus, experiments in this field cannot be designed, and the data properly interpreted, without an analytical description of the interaction of optical radiation with irregular surfaces.

Until recently, the theory of scattering from rough surfaces was severly limited, even in the geometrical-optics realm appropriate to optical studies. This was due to an incomplete treatment of the geometrical-optics description, in which the nonlocal effects of shadowing and multiple scatter of the rays

were neglected. This neglect was manifested in the inability of the theory to conserve energy.[3] An extended scattering theory now exists[4] which accounts for shadowing and double-scatter effects in a fully consistent way, i.e., energy is conserved to a high degree of accuracy. While all the tools necessary for an accurate determination of the scattered skylight are now at hand, there has been no corresponding effort directed towards a theory of rough-surface transmission. In this report, we present a complete geometrical-optics theory of transmission, with shadowing and multiple-reflection effects entering the description in a natural and consistent way. There are no restrictions on incidence or observation angles, and the full range of ocean sea states can be treated accurately.

We treat the idealized problem of two different semi-infinite and homogeneous media separated by a normally distributed rough boundary. A source is located in one medium, and the observer is located in the other. In Section II, we define the theoretical quantities of interest, namely the transmission coefficient which describes the angular radiation pattern after refraction, and the absorptivity, which gives the total fraction of incident energy entering the observer medium. In Section III, a formal analysis of the absorptivity is presented in terms of the physical subprocesses into which energy may be channeled. Thus, the dominant process would be simple absorption of an incident ray, but, for a rough boundary, we can also have absorption followed by reflection below the surface, as well as multiple reflection above the surface followed by absorption. The description is further complicated by the small, but nonzero, probability that a light ray may be transmitted into an undulation of the observer medium only to be refracted back into the source medium. A consistent approach to the cumulative effects of all the various higher-order processes is presented.

Section IV contains an analogous, and very formal, representation of the second theoretical quantity, the transmission coefficient, in terms of physical subprocesses. The first actual application of the transmission theory to random rough surfaces is carried through in Section V. Specifically, we derive the direct transmission coefficient which describes the angular transmitted radiation pattern in the absence of multiple-reflection contributions. Shadowing corrections are incorporated automatically. A simple exponential model is used

to account for propagation losses. The result is applicable to the full two dimensional rough surface and, as such, it contains all polarization information. Not surprisingly, the strength of the transmission at a given observation angle is proportional to the probability density of that slope which connects the incident and observation directions by Snell's law. It is of interest that the direct transmission coefficient is identically zero for some ranges of observation angle, regardless of the rms surface slope.

The results of a shadowing theory tailored to the transmission problem are merely cited in Section V, but in Section VI we present a full analytical treatment. The illumination probabilities derived here are not trivial extensions of the illumination probabilities used in scattering theory. This follows because the source and observer are now on opposite sides of the rough boundary, so that a high point on the surface relative to the source is a low point on the surface relative to the observer. The results are obtained in terms of gamma functions, which are fully tabulated.

Higher-order transmission terms are formulated in Sections VII and VIII for the mathematically tractable one dimensional rough surface (cylindrical symmetry). Thus, a ray of sunlight may reflect from one point on the ocean surface only to intersect the surface again at another point. There will be transmission across the surface boundary at the second point as well as direct transmission at the first scatter point. The angular profile due to transmission at the second point fills all possible observation angles, but the total fraction of incident energy associated with this double-scatter process is estimated at only about 3% of the directly transmitted energy for likely ocean roughnesses. The corresponding process, in which a directly transmitted ray of sunlight is internally reflected by the rough boundary is also formulated. It is expected to contribute significantly to the below-surface radiation profile only for near-grazing incidence and the very roughest of ocean surfaces.

In Section IX, we consider the inverse problem of refraction into the medium of lower index (e.g., water-to-air). Fortunately, it is not necessary to repeat the previous calculations, but instead we apply the principle of reciprocity. This theorem relates the effects of an interchange of source and observer, and so is ideally suited for application to the inverse problem.

Finally, in Section X, we derive the formal relation between observed intensity and the transmission and scattering coefficients which must be used in any analytical treatment of sea color. A homogeneous sea with flat bottom (of zero slope) is considered as a special case.

II. PROBLEM STATEMENT AND DEFINITIONS

We consider the idealized situation of two different homogeneous and semi-infinite media separated by an irregular boundary (Fig. 1). One of the media contains a source, with the receiver located in the second medium. For example, sunlight is incident on a rough ocean surface and we are interested in the angular distribution of transmitted radiation below the surface. We will take the index of refraction of the source medium as unity, whereas the second medium will be represented everywhere by a complex constant $\tilde{n}$. The homogeneity assumption means that once clear of the surface, the refracted rays propagate rectilinearly until completely absorbed. A simple exponential absorption is associated with each ray. We will also restrict the investigation to those cases where the imaginary part of the refractive index is small, e.g., the problem of optical transmission into clear water. Then Snell's law relates, with high accuracy, the angle of refraction to n, the real part of $\tilde{n}$ (Appendix A).

There are two theoretical quantities of interest to us. The first is the absorptivity, defined as the fraction of incident energy which is not scattered. The absorptivity is a function only of the angle of incidence. Now, energy crossing the boundary must propagate some distance into the medium (of index $\tilde{n}$) before it is absorbed. This view leads immediately to the second theoretical quantity, the transmission coefficient, which determines the fraction of incident energy observed at some angle relative to the (-z) axis and at a depth ℓ. Thus, the transmission coefficient describes the angular spreading, due to surface roughness, of the refracted radiation, and it also accounts for propagation losses. If we designate $a(\theta_0,\phi_0)$ as the absorptivity due to a source at (θ_0,ϕ_0), and if $t(\theta,\phi;\ \theta_0,\phi_0;\ K\ell)$ is the transmission coefficient for an observer at (θ,ϕ) and at depth ℓ in a medium with absorption coefficient K, then

$$a(\theta_0,\phi_0) = \int d\Omega\ t(\theta,\phi;\ \theta_0,\phi_0;\ K\ell=0)\ , \tag{2.1}$$

where the integration is over the lower hemisphere (all possible refracted angles).

III. PARTIAL-SURFACE REPRESENTATION FOR ABSORPTIVITY

Radiation incident on a perfectly flat surface is refracted according to Snell's law. For a rough ocean surface and optical radiation, we can apply Snell's law to each of the locally flat surface elements comprising the boundary. Optical wavelengths will be small compared with even capillary structure on the ocean surface, so a ray theory of transmission can be expected to give very accurate results.

A ray theory of transmission for an irregular surface is actually the sum of many processes. The most important are illustrated by Figs. 2-4. Every incident ray contributes to the direct transmission process, $\vec{k}_0 \rightarrow \vec{k}_{1T}$, (Fig. 2). Figure 3 illustrates an additional transmission contribution, $\vec{k}_0 \rightarrow \vec{k}_1 \rightarrow \vec{k}_{2T}$, arising from the reflection of the incident ray at point 1 and a subsequent intersection with the surface at point 2. The importance of this process increases with increasing rms surface slope. The associated process of a transmitted ray which undergoes a subsequent reflection (below the surface) at point 2, $\vec{k}_0 \rightarrow \vec{k}_{1T} \rightarrow \vec{k}_{2R}$, is shown in Fig. 4. A correct theory of transmission must account for all such diagrams.

In this section we will restrict the quantitative discussion to the special one-dimensional rough surface (cylindrical symmetry) in order to emphasize the physical picture; cross polarization complications are introduced in Section V. Thus, we consider radiation of unit intensity and beam width S_0 incident on an irregular surface described by $z = \zeta(x,y) = \zeta(x)$. The incident, transmitted, and scattered rays are all contained in the x=z plane. The optical wavelength is small compared to all surface parameters. The scattering and transmission processes are incoherent in this geometrical-optics realm as each surface element simply obeys Snell's law, independent of any other surface element.

We imagine a ray trace for each surface element, and we will include that element in one of a number of classes, depending on the scattering character of that trace. Specifically, we divide the total surface into two parts, $\Sigma = \Omega + \Omega'$, where every surface element in Ω is visible to the incident beam with direction $\vec{k}_0$. The elements in Ω' are shielded from the incident beam, and as their contributions are multiplied by zero incident intensity, their ray traces have zero weight. The illuminated subset is now split into two parts, $\Omega = \Omega_1 + \Omega_1'$, where every element in Ω_1 is characterized by a specularly scattered ray $\vec{k}_1$ which does not intersect the surface elsewhere (Fig. 2). For every element of the subset $\Omega_1' = \Omega_2 + \Omega_2'$, the ray $\vec{k}_1$ does intersect the surface

again, but now Ω_2 is defined as a smaller subset with the property that the follow-up ray $\vec{k}_2$ does not intersect the surface elsewhere (Fig. 3). We continue this bookkeeping until we achieve an empty subset, say Ω_m'' (a maximum of m surface scatterings):

$$\Omega = \sum_{l=1}^{m} \Omega_l \quad . \tag{3.1}$$

The incident power intercepted by a surface element dS_1 in Ω is unit intensity times the elemental area projected onto the incident wave front, i.e., $(-\hat{k}_0 \cdot \hat{n}_1)dS_1$. The initial polarization is taken as either vertical or horizontal, and the scattered or transmitted radiation will retain this polarization because of the cylindrical symmetry. If the reflectivity at surface point i is r_i (for either vertical or horizontal polarization), then $(1-r_i)$ is the fraction of energy incident at this point which is transmitted. We are now in a position to write down the quantity $T(\theta_0)$, defined as the total fraction of energy incident from $\vec{k}_0$ which is refracted across the boundary:

$$\begin{aligned} S_0 T(\theta_0) &= S_0 \sum_{i=1}^{m} T_i(\theta_0) \\ &= \int_{\Omega} dS_1(\cos\alpha_1)[1-r_1(\cos\alpha_1)] \\ &+ \int_{\Omega_1'} dS_1(\cos\alpha_1) r_1(\cos\alpha_1)[1-r_2(\cos\alpha_2)] \\ &+ \ldots \int_{\Omega_{m-1}'} dS_1(\cos\alpha_1) r_1 r_2 \ldots r_{m-1}(1-r_m) \quad , \end{aligned} \tag{3.2}$$

where $(-\hat{k}_0 \cdot \hat{n}_1) = \cos\alpha_1$, $(-\hat{k}_1 \cdot \hat{n}_2) = \cos\alpha_2$, and an obvious abbreviation for the arguments of the reflectivities is introduced in the m'th term. Each integration is taken over the indicated subset of the total surface. Equation (3.2) accounts for all orders of multiple reflection in the source medium. For example, the contribution to $T(\theta_0)$ at point 1 in Fig. 3 is contained in the first term, while the energy crossing the boundary at point 2 of the same diagram is accounted for in the second term. Note, the quantity S_0 is the normalization factor, for when r = 0 and T = 1, we must have

$$S_0 = \int_{\Omega} dS_1(-\hat{k}_c \cdot \hat{n}_1) \quad . \tag{3.3}$$

The absorptivity is defined as the fraction of incident energy which is not scattered. Unfortunately, the quantity $T(\theta_0)$ is not the absorptivity. This follows because energy refracted across a rough-surface interface can be refracted back into the source medium, there to scatter and refract again. Figure 4 contains an illustration of this point. Thus, an incident ray refracts at point 1, then propagates internally to point 2 where it reflects and may refract (depending on the slope at point 2) back into the source medium. The doubly refracted ray $\vec{k}_{2TT}$ is always directed towards the surface, so it will be reflected and refracted again at point 3. The double transmission plus reflection process augments the scattered radiation pattern. Therefore, T is greater than the absorptivity by an additional scatter contribution $\delta(\theta_0)$:

$$T(\theta_0) \;=\; a(\theta_0) + \delta(\theta_0) \quad . \tag{3.4}$$

In order to obtain a partial-surface expansion for $a(\theta_0)$ alone, we must expand each of the T_i in Eq. (3.2) ı a partial-surface representation. We will simply state in passing that, while $T(\theta_0)$ is not strictly the fundamental quantity of interest, it may be bounded in value from both above and below. As $\delta(\theta_0)$ is expected to be very small in magnitude, this can be an important advantage in numerical computations of the absorptivity.

We now expand T_1 in a partial-surface series which accounts for all the possible processes the ray may undergo after refraction at point 1. We have, explicitly,

$$S_0 T_1(\theta_0) \;=\; \int_\Omega dS_1 \cos\alpha_1 (1-r_1) \quad , \tag{3.5}$$

and we define

$$\begin{aligned} \Omega &= \Omega_T + \Omega_T' \quad , \\ \Omega_T' &= \Omega_{TR} + \Omega_{TR}' \; . \end{aligned} \tag{3.6}$$

Here, Ω_T contains those surface elements in Ω such that the refracted ray $\vec{k}_{1T}$ does not intersect the surface again (Fig. 2), whereas the ray $\vec{k}_{1T}$ does intersect the surface again for elements of surface in the subset Ω_T' (Fig. 4). The further decomposition of Ω_T' depends on whether the ray reflected from point 2 in Fig. 4 clears the surface (Ω_{TR}) or intercepts it again (Ω_{TR}').

We can write, with these definitions,

$$S_0T_1(\theta_0) = \int_{\Omega_T} dS_1(\cos\alpha_1)(1-r_1) + \int_{\Omega_T'} dS_1(\cos\alpha_1)(1-r_1)[r_2+(1-r_2)]$$

$$= S_0[a_1(\theta_0) + a_{1R}(\theta_0) + \Delta_1(\theta_0)] \quad , \tag{3.7}$$

where

$$S_0a_1(\theta_0) = \int_{\Omega_T} dS_1(\cos\alpha_1)(1-r_1) \quad , \tag{3.8}$$

$$S_0a_{1R}(\theta_0) = \int_{\Omega_{TR}} dS_1(\cos\alpha_1)(1-r_1)r_2 \quad , \tag{3.9}$$

$$S_0\Delta_1(\theta_0) = \int_{\Omega_{TR}'} dS_1(\cos\alpha_1)(1-r_1)r_2 + \int_{\Omega_T'} dS_1\cos\alpha_1(1-r_1)(1-r_2) \tag{3.10}$$

The quantities a_1 and a_{1R} are true contributions to the absorptivity, for Ω_T and Ω_{TR} were constructed such that k_{1T} and k_{2R}, respectively, do not intercept the surface again. On the other hand, Δ_1 contains higher-order absorption processes as well as contributions to the scattering diagram. For example, the second term in Eq. (3.10) is the double refraction process illustrated in Fig. 4; some of the energy associated with this term goes into absorption and some into scattering.

A most important aspect to the expansion of Eq. (3.5) is that the value of the residue term, $\Delta_1(\theta_0)$, can be calculated indirectly. This is true because $T_1(\theta_0)$ is nothing more than the direct emissivity[5] of the rough surface. Thus, a computer program already exists for the evaluation of $T_1(\theta_0)$. Some results are listed in Tables I and II for a rough water surface and incident radiation of wavelength 0.7μ. In addition, both a_1 and a_{1R} can be calculated numerically (although we have not yet done this), so we can solve for Δ_1 by use of Eq. (3.7). Alternatively, we can limit numerical computations to T_1 and a_1 and solve for the value of $(a_{1R}+\Delta_1)$. This will ordinarily be sufficient as a_{1R} is surely negligible compared to a_1 save for near grazing incidence and very rough surfaces (Section VIII).

Table I

The quantities $T_1(\theta_0)$ and $T_2(\theta_0)$, evaluated for a rough water surface of 15° rms slope and horizontally polarized incident radiation of wavelength 0.7μ.

θ_0	$[T_1(\theta_0)] \times 293$	$[T_2(\theta_0)] \times 293$	$T_2/T_1 \times 100$
30°	283.0	0.47	0.17
50	271.4	3.03	1.11
65	252.4	7.32	2.90
75	235.0	7.24	3.08
80	225.5	7.36	3.26
85	215.4	4.11	1.91

Table II

The quantities $T_1(\theta_0)$ and $T_2(\theta_0)$, evaluated as in Table I, but for vertically polarized incident radiation.

θ_0	$[T_1(\theta_0)] \times 293$	$[T_2(\theta_0)] \times 293$	$T_2/T_1 \times 100$
30°	288.3	0.12	0.04
50	287.6	1.53	0.53
65	279.9	5.23	1.87
75	270.1	6.47	2.40
80	264.2	7.14	2.70
85	256.9	4.29	1.67

We simply write down the analogous results for $T_2(\theta_o)$:

$$S_o T_2(\theta_o) = \int_{\Omega_1'} dS_1(\cos\alpha_1) r_1(1-r_2)$$

$$= S_o[a_2(\theta_o) + \Delta_2(\theta_o)] \quad , \tag{3.11}$$

where

$$S_o a_2(\theta_o) = \int_{(\Omega_1')_T} dS_1(\cos\alpha_1)\, r_1(1-r_2) \quad , \tag{3.12}$$

and the residue term, $\Delta_2(\theta_o)$, contains higher-order absorption and scatter processes. The subset $\Omega_1'\epsilon\Omega$ has been further subdivided, $\Omega_1' = (\Omega_1')_T + (\Omega_1')_T'$, such that the ray $\vec{k}_{2T}$ either clears the surface $[(\Omega_1')_T]$ or strikes the surface again $[(\Omega_1')_T']$. The quantity $T_2(\theta_o)$ is the emission plus reflection term[5] in the lower-bound calculation of rough-surface emissivity and, again, a computer program is available for its evaluation (Tables I and II). The quantity $a_2(\theta_o)$ is a contribution to the absorptivity, by construction, and it can be calculated numerically. Thus $\Delta_2(\theta_o)$ can be calculated from Eq. (3.11). We neglect the remaining terms $T_3, \ldots T_m$ relative to T_1 and T_2; this is a valid approximation for the rms surface slopes appropriate to the ocean surface.

In summary, we note that $T_1, T_2, \ldots T_m$ can be thought of as channels through which the incident energy is funneled into an infinite number of processes. For example, the double transmission process of Fig. 4 is funded by the energy budget associated with T_1. It is not necessary to evaluate the magnitude of these higher-order processes on an individual basis, but, instead, it is meaningful to lump them together in the Δ_i. For example, the fraction of the incident energy funneled into channels T_1 and T_2 can be calculated, so Δ_1 and Δ_2 can be determined. We will simply assume here that the higher-order absorption processes in Δ_1, Δ_2 (and $T_3, \ldots T_m$) are negligible compared to the smallest of (a_1, a_{1R}, a_2) for the rms slopes associated with the ocean surface. Then, a meaningful expansion for the absorptivity is

$$a(\theta_o) = a_1(\theta_o) + a_{1R}(\theta_o) + a_2(\theta_o) + \ldots \quad , \tag{3.13}$$

with (a_1, a_{1R}, a_2) defined formally by Eqs. (3.8), (3.9), and (3.12) respectively.

In passing, we note that every term in the T equation is positive. The technique of this section was to further subdivide each of the T_i into a series of positive terms, some of the elements to be grouped under $a(\theta_0)$ and the rest under $\delta(\theta_0)$ [Eq.(3.4)]. The net result is that each term on the right-hand side of Eq. (3.13) is positive. Thus, by taking just the first m terms as an approximation, we have a lower bound to the absorptivity.

IV. PARTIAL-SURFACE REPRESENTATION FOR THE TRANSMISSION COEFFICIENT

The absorptivity provides no knowledge of the angular distribution of transmitted energy. We define the transmissivity, $d\Omega\, t(\vec{k},\vec{k}_0;\ell)$ as the fraction of energy incident from $\vec{k}_0$ which is refracted into solid angle $d\Omega$ about the observer direction $\vec{k}$, for an observer at depth ℓ (measured from the mean surface height). It is conventional to deal simply with t, the transmission coefficient. As with the absorptivity, there is a partial-surface expansion for t. Indeed, as a consequence of ray theory, the partial transmission coefficients, t_1, t_{1R}, t_2 are, at least formally, simple extensions of the corresponding a_1, a_{1R}, a_2 of Section III.

We will first detail the derivation of the direct transmission coefficient, t_1, for a one dimensional rough surface. The appropriate partial surface is Ω_T; i.e., every surface element dS_1 in Ω_T is visible to the source and to the observer. The power incident on dS_1 is $(\cos\alpha_1)dS_1$, and the fraction transmitted is $dS_1(\cos\alpha_1)[1-r_1(\cos\alpha_1)]$. Since $dS_1\varepsilon\Omega_T$, every refracted ray clears the (under) surface. We account for the absorption along the propagation path of the ray by a multiplicative exponential absorption factor, $\exp[-K(\ell+z_1)\sec\theta_{1T}]$, where K is the absorption coefficient, z_1 is the surface height of point 1, and θ_{1T} is the angle of propagation for the refracted ray. We must also multiply by the probability that the ray reaches the observer stationed at angle θ; because of the ray optics and the assumed homogeneity of the medium, the probability density is simply a delta function $\delta(\theta-\theta_{1T})$. Finally, we sum over all $dS_1\varepsilon\Omega_T$:

$$S_0 t_1(\theta,\theta_0) = \int_{\Omega_T} dS_1(\cos\alpha_1)[1-r_1(\cos\alpha_1)]\exp[-K(\ell+z_1)\sec\theta_{1T}]\delta(\theta-\theta_{1T}) \ , \qquad (4.1)$$

where S_0, defined in Eq.(3.3), is included on the left-hand side for normalization.

The analogous expression for the direct transmission coefficient applicable to a general, two dimensional, random rough surface is

$$S_o t_1^{ij}(\vec{k},\vec{k}_o) = \int_{\Omega_T} dS_1(\cos\alpha_1)(\tau^{ij})\exp[-K(\ell+z_1)\sec\theta_{1T}]\delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T}) \quad . \tag{4.2}$$

Here, the superscripts reflect the polarization character of the problem. Specifically, the quantity $t^{ij}(\vec{k},\vec{k}_o)$ determines the fraction of the incident radiation from $\vec{k}_o$, in polarization state i, which is transmitted into an infinitesimal solid angle about observation direction $\vec{k}$, with final polarization state j. The fraction of incident power, with polarization state i, transmitted across an arbitrarily oriented plane surface into final polarization state j is represented by τ^{ij}; for cylindrical symmetry the incident polarization is retained after refraction and τ^{ij} reduces to $(1-r^i)\delta_{ij}$. The general form of τ^{ij} is derived in Appendix B. Equation (4.2) will usually be the major component of the exact transmission coefficient.

We will simply state the results for $\bar{t}_{1R}$ and $\bar{t}_2$, and we will restrict consideration to one dimensional roughness:

$$\begin{aligned} S_o t_{1R}(\theta,\theta_o) = & \int_{\Omega_{TR}} dS_1(\cos\alpha_1)(1-r_1)\exp[-K(z_1-z_2)\sec\theta_{1T}] \\ & \times r_2 \exp[-K(\ell+z_2)\sec\theta_{2R}] \\ & \times \delta(\theta-\theta_{2R}) \quad , \end{aligned} \tag{4.3}$$

$$\begin{aligned} S_o t_2(\theta,\theta_o) = & \int_{(\Omega_1')_T} dS_1(\cos\alpha_1) r_1(1-r_2) \exp[-K(\ell+z_2)\sec\theta_{2T}] \\ & \times \delta(\theta-\theta_{2T}) \end{aligned} \tag{4.4}$$

The series for the transmission coefficient has a form analogous to Eq. (3.13):

$$t(\theta,\theta_o) = t_1(\theta,\theta_o) + t_{1R}(\theta,\theta_o) + t_2(\theta,\theta_o) + \ldots \tag{4.5}$$

Each term in Eq. (4.5) is positive, so the retention of the first m terms provides a lower bound to $t(\theta,\theta_o)$.

V. DIRECT TRANSMISSION COEFFICIENT FOR RANDOM ROUGH SURFACES

The formalism of Section IV is especially suitable for application to a random rough surface. Thus, we will view $\zeta(x,y)$ as one of an ensemble of possible surfaces generated by a stationary random process and calculate the ensemble average of the transmission coefficient. The averaging process replaces the deterministic surface classes by known probability functions, and it also provides for an integration over the delta-function factors.

We will consider only the direct transmission contribution $t_1^{ij}(\vec{k},\vec{k}_0)$ in this section. We can convert the integration over Ω_T into an integration over the entire surface Σ by defining two functions which take on only the values of zero and unity. Thus, we define an illumination function $\varepsilon(x,y;\vec{k}_0)$, which has value unity if the surface point at (x,y) is illuminated from direction $\vec{k}_0$ and which has value zero if that point is shadowed. In addition, we define $\nu(x,y;\vec{k}_{1T})$ to have value unity if the transmitted ray which leaves point (x,y) in direction $\vec{k}_{1T}$ intersects the surface at some other point, whereas it has value zero if $\vec{k}_{1T}$ does not intersect the surface again. With these definitions, Eq. (4.2) becomes

$$
\begin{aligned}
S_0 t_1^{ij}(\vec{k},\vec{k}_0) = \int_{\Sigma} & dS_1 (\cos\alpha_1)\tau^{ij} \exp[-K(\ell+z_1)\sec\theta_{1T}] \\
& \times \ \varepsilon(x,y;\vec{k}_0)[1-\nu(x,y;\vec{k}_{1T})] \\
& \times \ \delta(\theta-\theta_{1T})\ \delta(\phi-\phi_{1T}) \ . \qquad (5.1)
\end{aligned}
$$

Now, the coordinates of the transmitted ray for surface element dS_1 are (θ_{1T},ϕ_{1T}), and by Snell's law we have $\theta_{1T} = \theta_{1T}(\theta_0,s_{1x},s_{1y})$ and $\phi_{1T} = \phi_{1T}(\theta_0,s_{1x},s_{1y})$. Here, (s_{1x},s_{1y}) are the components of surface slope at point 1. The surface element is $dS_1 = (1+s_{1x}^2 + s_{1y}^2)^{\frac{1}{2}}\, dx_1 dy_1$, and $\cos\alpha_1$ and τ^{ij} depend only on (x_1,y_1) through the slope (s_{1x},s_{1y}). Therefore, the integrand in Eq. (5.1) depends on (x_1,y_1) only through $z_1(x_1,y_1)$, $s_{1x}(x_1,y_1)$, $s_{1y}(x_1,y_1)$, $\varepsilon(x_1,y_1)$, $\nu(x_1,y_1)$. For any stationary random process, there are no preferred points, so the appropriate probability distribution is independent of (x_1,y_1). Thus, the averaging procedure leaves a trivial coordinate integration:

$$S_o \left\langle t_1^{ij}(\vec{k},\vec{k}_o)\right\rangle = \int_{-L}^{L}dx_1\int_{-L'}^{L'}dy_1\int_{-\infty}^{\infty}dz_1\int_{-\infty}^{\infty}ds_{1x}\int_{-\infty}^{\infty}ds_{1y} \sum_{\varepsilon_1=0}^{1}\sum_{\nu_1=0}^{1} P(z_1,s_{1x},s_{1y},\varepsilon_1,\nu_1)$$

$$\times\ \varepsilon_1(1-\nu_1)f_1^{ij}(s_{1x},s_{1y})$$

$$\times\ \exp[-K(\ell+z_1)\sec\theta_{1T})]$$

$$\times\ \delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T})$$

$$= (2L)(2L')\int_{-\infty}^{\infty}dz_1\int_{-\infty}^{\infty}ds_{1x}\int_{-\infty}^{\infty}ds_{1y}\ P[z_1,s_{1x},s_{1y},\varepsilon_1(\vec{k}_o) = 1,\nu_1(\vec{k}_{1T}) = 0]$$

$$\times\ f_1^{ij}\ \exp[-K(\ell+z_1)\sec\theta_{1T}]\delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T})\ , \tag{5.2}$$

where

$$f_1^{ij}(s_{1x},s_{1y}) = (1+s_{1x}^2+s_{1y}^2)^{\frac{1}{2}}\ (\cos\alpha_1)\ \tau^{ij}(s_{1x},s_{1y})\ . \tag{5.3}$$

The segments along the x and y axes cut out by the incident beam are taken here as 2L and 2L', respectively, so that, in these terms, $S_o = (2L)(2L')\cos\theta_o$). Also, we note that the ray $\vec{k}_{1T}$ must intersect the surface if $\theta_{1T} > \pi/2$ (Fig. 2). For $\theta_{1T} < \pi/2$, the probability that the ray $\vec{k}_{1T}$ does not intersect the surface elsewhere is equivalent mathematically to the probability that the point (x_1,y_1) is illuminated (by an imaginary source below the surface) from the direction $-\vec{k}_{1T}$. On the basis of these remarks, Eq. (5.2) can be expressed in the following form:

$$\left\langle t_1^{ij}(\vec{k},\vec{k}_o)\right\rangle = (\sec\theta_o)\int_{-\infty}^{\infty}dz_1\int_{-\infty}^{\infty}ds_{1x}\int_{-\infty}^{\infty}ds_{1y}\; P[z_1,s_{1x},s_{1y},\ \varepsilon_1(\vec{k}_o,-\vec{k}_{1T}) = 1]$$

$$\times\ u(\pi/2-\theta_{1T})\ f_1^{ij}(s_{1x},s_{1y})$$

$$\times\ \exp[-K(\ell+z_1)\sec\theta_{1T}]$$

$$\times\ \delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T}) \quad , \tag{5.4}$$

where u is a step function with value unity for positive argument and value zero for negative argument.

The above multiple integral is quite complicated because of the integration over height z_1. It is sensible to approximate the factor z_1 in the exponential argument by the average value of the "illuminated" surface height, ℓ_1, defined by

$$\ell_1(\theta_o,\theta_{1T}) \equiv \int_{-\infty}^{\infty}dz_1\ z_1\ P[z_1|\varepsilon_1(\vec{k}_o,-\vec{k}_{1T}) = 1] \ , \tag{5.5}$$

where the probability density of heights is conditional on the illumination of point 1 from both the $\vec{k}_o$ and $-\vec{k}_{1T}$ directions. The gross effect of absorption is therefore retained while the analysis is greatly simplified. The presence of the delta function permits us to replace θ_{1T} by θ, so

$$\left\langle t_1^{ij}(\vec{k},\vec{k}_o)\right\rangle = (\sec\theta_o)\exp[-K(\ell+\ell_1)\sec\theta]$$

$$\times\ \int_{-\infty}^{\infty}dz_1\int_{-\infty}^{\infty}ds_{1x}\int_{-\infty}^{\infty}ds_{1y}P[z_1,s_{1x},s_{1y},\varepsilon_1(\vec{k}_o,-\vec{k}_{1T}) = 1]$$

$$\times\ f_1^{ij}(s_{1x},s_{1y})u(\pi/2-\theta_{1T})$$

$$\times\ \delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T}) \quad ,$$

$$\left\langle t_1^{ij}(\vec{k},\vec{k}_0)\right\rangle = (\sec\theta_0)\exp[-K(\ell+\kappa_1)\sec\theta]$$

$$\times \int_{-\infty}^{\infty} ds_{1x} \int_{-\infty}^{\infty} ds_{1y}\ P[s_{1x},s_{1y},\epsilon_1(\vec{k}_0,-\vec{k}_{1T}) = 1]u(\pi/2-\theta_{1T})$$

$$\times\ f_1^{ij}(s_{1x},s_{1y})\delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T}) \quad , \qquad (5.6)$$

where the last step follows from the assumed normalization of the probability density.

The remaining integrations can be carried out by a change of variables from (s_{1x},s_{1y}) to (θ_{1T},ϕ_{1T}). The equations of transformation are, from Appendix C,

$$s_{1x} = \frac{(\sin\theta_0 - n\sin\theta_{1T}\cos\phi_{1T})}{(n\cos\theta_{1T} - \cos\theta_0)} \quad ,$$

$$s_{1y} = \frac{-n\sin\theta_{1T}\sin\phi_{1T}}{(n\cos\theta_{1T} - \cos\theta_0)} \quad , \qquad (5.7)$$

and the Jacobian of the transformation is

$$J(\theta_{1T},\phi_{1T}) = \frac{n^2\sin\theta_{1T}[n-(\hat{k}_0\cdot\hat{k}_{1T})]}{(n\cos\theta_{1T}-\cos\theta_0)^3} \quad . \qquad (5.8)$$

Here, n is the real part of the index of refraction. One must be very careful at this stage because the integration limits on the (θ_{1T},ϕ_{1T}) integrals are not simply $0 < \theta_{1T} < \pi/2$, $0 < \phi_{1T} < 2\pi$. This follows because the orientation of a plane surface which connects the desired direction for transmission with the given direction of incidence may not be allowed on physical grounds. More specifically, there are two conditions on (θ_{1T},ϕ_{1T}) which follow from physical constraints on this problem. First, we require that $\hat{n}_1\cdot\hat{z} > 0$ everywhere, i.e., the angle between any surface normal and the z direction must be less than 90^o. This restriction arises because we are assuming a single-valued normally distributed surface, a property of which is finite slopes everywhere. But, from

Appendix C, we see that this condition implies

$$\cos\theta_{1T} > \frac{\cos\theta_o}{n} \quad . \tag{5.9}$$

A second condition is the illumination requirement $(-\hat{k}_o \cdot \hat{n}_1) > 0$; i.e., a surface point cannot be illuminated by the incident ray unless the angle between the normal to the point and the direction of illumination is less than 90°. This obvious condition imposes a very severe restriction on (θ_{1T},ϕ_{1T}):

$$\hat{k}_o \cdot \hat{k}_{1T} > \frac{1}{n} \quad . \tag{5.10}$$

The simple way to incorporate these two restrictions on the transmitted direction is by use of two more step functions, $u\left(\cos\theta_{1T} - \frac{\cos\theta_o}{n}\right)$ and $u(\hat{k}_o \cdot \hat{k}_{1T} - \frac{1}{n})$; i.e.,

$$\int_{-\infty}^{\infty} ds_{1x} \int_{-\infty}^{\infty} ds_{1y}\, u(\pi/2 - \theta_{1T}) \rightarrow \int_0^{\pi/2} d\theta_{1T} \int_0^{2\pi} d\phi_{1T}\, J(\theta_{1T},\phi_{1T})\, u\left(\cos\theta_{1T} - \frac{\cos\theta_o}{n}\right)$$
$$\times\ u\left(\hat{k}_o \cdot \hat{k}_{1T} - \frac{1}{n}\right) . \tag{5.11}$$

The step functions automatically impose the proper integration ranges.

The application of Eqs.(5.7), (5.8), and (5.11) to Eq.(5.6) yields

$$t_1^{ij}(\vec{k},\vec{k}_o) = (\sec\theta_o)\exp[-K(\ell+\ell_1)\sec\theta]$$
$$\times \int_0^{\pi/2} d\theta_{1T} \int_0^{2\pi} d\phi_{1T}\, J(\theta_{1T},\phi_{1T})\, u(n\cos\theta_{1T} - \cos\theta_o)u(\hat{k}_o \cdot \hat{k}_{1T} - \frac{1}{n})$$
$$\times\ P[s_{1x}(\theta_{1T},\phi_{1T}),\ s_{1y}(\theta_{1T},\phi_{1T}),\ \varepsilon_1(\vec{k}_o,-\vec{k}_{1T}) = 1]$$
$$\times\ f_1^{ij}\ \delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T}),$$

which simplifies immediately to

$$\left\langle t_1^{ij}(\vec{k},\vec{k}_0)\right\rangle = (\sec\theta_0)\{\exp[-K(\ell+\ell_1)\sec\theta]\}f_1^{ij}(s_{1x}^o,s_{1y}^o)$$

$$\times\ u(n\cos\theta-\cos\theta_0)u(\hat{k}_0\cdot\hat{k}-\frac{1}{n})$$

$$\times\ \frac{n^2[n-(\hat{k}_0\cdot\hat{k})]}{(n\cos\theta-\cos\theta_0)^3}\ P[s_{1x}^o,s_{1y}^o,\ \varepsilon_1(\vec{k}_0,-\vec{k}) = 1] \qquad (5.12)$$

where we have used the properties of the delta function and the normalization

$$\int_0^{\pi/2} d\theta_{1T}\int_0^{2\pi} d\phi_{1T}\ \sin\theta_{1T}\ \delta(\theta-\theta_{1T})\delta(\phi-\phi_{1T}) = 1 .$$

Geometrical quantities in Eq.(5.12) which need to be identified explicitly are

$$s_{1x}^o = \frac{(\sin\theta_0-n\sin\theta\cos\phi)}{(n\cos\theta-\cos\theta_0)} ,$$

$$s_{1y}^o = \frac{-n\sin\theta\sin\phi}{(n\cos\theta-\cos\theta_0)} , \qquad (5.13)$$

and

$$f_1^{ij}(s_{1x}^o,s_{1y}^o) = (1+s_{1x}^{o2}+s_{1y}^{o2})^{\frac{1}{2}}\ (\cos\alpha_1^o)\tau^{ij}(s_{1x}^o,s_{1y}^o) , \qquad (5.14)$$

whence, after some algebra,

$$(1+s_{1x}^{o2}+s_{1y}^{o2})^{\frac{1}{2}} = \frac{[n^2+1-2n(\hat{k}_0\cdot\hat{k})]^{\frac{1}{2}}}{(n\cos\theta-\cos\theta_0)} , \qquad (5.15)$$

$$\cos\alpha_1^o = \frac{n[(\hat{k}_0\cdot\hat{k})-\frac{1}{n}]}{[n^2+1-2n(\hat{k}_0\cdot\hat{k})]^{\frac{1}{2}}} . \qquad (5.16)$$

From Appendix B, the fraction of energy transmitted across the properly oriented plane surface, $\tau^{ij}(s_{1x}^o,s_{1y}^o)$, is a function of a_1^o, b_1^o, c_1^o, t_H^o, t_V^o, $\cos\alpha_1^o$ and $\cos\alpha_1'(s_{1x}^o,s_{1y}^o)$. The following list specifies each of these quantities in terms of known factors:

$$\cos\alpha_1'(s_{1x}^0,s_{1y}^0) = \frac{[(n^2-1)+\cos^2\alpha_1^0]^{\frac{1}{2}}}{n} \quad , \tag{5.17}$$

$$a_1^0 = \frac{-\hat{k}\cdot\hat{h}_0}{|\hat{k}_0 x\hat{k}|} \quad ; \quad \hat{h}_0 = \hat{y} \quad , \tag{5.18}$$

$$b_1^0 = \frac{-\hat{k}\cdot\hat{v}_0}{|\hat{k}_0 x\ \hat{k}|} \quad ; \quad \hat{v}_0 = -\cos\theta_0\ \hat{x} + \sin\theta_0\hat{z} \quad , \tag{5.19}$$

$$c_1^0 = \frac{-(\hat{k}_0 x\ \hat{k})\cdot(\hat{k}\ x\ \hat{z})}{(|\hat{k}_0 x\ \hat{k}|)(|\hat{k}\ x\ \hat{z}|)} \quad , \tag{5.20}$$

$$c_2^0 = \frac{-(\hat{k}_0 x\ \hat{k})\cdot[\hat{z}-(\hat{k}\cdot\hat{z})\hat{k}]}{(|\hat{k}_0 x\ \hat{k}|)(|\hat{k}\ x\ \hat{z}|)} \quad , \tag{5.21}$$

$$t_H^0 = 1 + R_H(\cos\alpha_1^0) \quad , \tag{5.22}$$

$$t_V^0 = \frac{1+R_V(\cos\alpha_1^0)}{n} \quad . \tag{5.23}$$

The probability density has the form

$$P[s_{1x}^0,s_{1y}^0,\ \varepsilon_1(\vec{k}_0,-\vec{k}) = 1] = p(s_{1x}^0,s_{1y}^0)P[\varepsilon_1(\vec{k}_0,-\vec{k}) = 1|s_{1x}^0,s_{1y}^0] \ , \tag{5.24}$$

where, for the principal directions, the probability density of slopes is

$$p(s_{1x}^0,s_{1y}^0) = \frac{1}{2\pi}\left(\langle\zeta_x{}^2\rangle\langle\zeta_y^2\rangle\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{s_x^2}{\langle\zeta_x^2\rangle} + \frac{s_y^2}{\langle\zeta_y^2\rangle}\right)\right] \quad , \tag{5.25}$$

and, from Eq. (6.11) of the following section, the probability of illumination conditional on the slope value is

$$P[\varepsilon_1(\vec{k}_0,-\vec{k}) = 1|(s_{1x}^0,s_{1y}^0)] = \frac{1}{[1+2(B_0+B)]}\ \frac{\Gamma(1+2B_0)\Gamma(1+2B)}{\Gamma[1+2(B_0+B)]} \quad . \tag{5.26}$$

The quantities $\Gamma(x)$ are Gamma functions of argument x, and

$$B = [4(\pi)^{\frac{1}{2}}V]^{-1}[\exp(-V^2) - (\pi)^{\frac{1}{2}} V(1-\text{erf } V)] \quad , \tag{5.27}$$

with

$$V = \frac{\cot\theta}{[2(\langle\zeta_x^2\rangle\cos^2\phi + \langle\zeta_y^2\rangle\sin^2\phi)]^{\frac{1}{2}}} \tag{5.28}$$

The quantity B_0 follows from B by replacement of (θ,ϕ) by $(\theta_0,\phi_0 = 0)$. Equations (5.13) - (5.28) specify all quantities in Eq. (5.12) except ℓ_1, the average illuminated height in the exponential absorption factor. This factor is easily derived from the shadowing theory of Section VI:

$$\ell_1(\vec{k},\vec{k}_0) = (\pi/2)^{\frac{1}{2}} \sigma \frac{[B_0(\theta_0) - B(\theta,\phi)]}{(1+B_0+B)} \quad , \tag{5.29}$$

where σ is the rms height for the surface.

The numerous factors which we have just listed must not be allowed to obscure the overall simplicity of Eq. (5.12). Thus, there are no integrations to carry out; instead, the direct transmission is simply proportional to the probability density of that particular slope which connects $\vec{k}$ and $\vec{k}_0$ by Snell's law The probability that this characteristic slope is visible to both source and observer also enters. An additional feature of interest is the product of the two step functions, which can yield zero for some combinations of $\vec{k}$ and $\vec{k}_0$. Thus, it is not always possible to connect a direction of transmission with a given direction of incidence by any physically allowed plane-surface element. This is most easily demonstrated for the special case of the one dimensional rough surface. Then, the allowed range for the transmission angle θ is easily shown to be

$$\theta_0 - [\pi/2-\sin^{-1}(\tfrac{1}{n})] < \theta < \pi/2-\sin^{-1}\left(\frac{\cos\theta_0}{n}\right) \quad . \tag{5.30}$$

For example, if the angle of incidence is $\theta_0 = 20^0$, then there is no direct transmission for $\theta < 21^0$ and $\theta > 45^0$. Of course, the higher-order processes, t_{1R} and t_2, will contribute something to these "null" ranges.

VI. SHADOWING PROBABILITIES

An element of surface has only a certain probability of being illuminated by radiation incident from directions other than normal. For example, the angle between the surface normal and the direction of incidence may exceed 90^0, so that the slope of the element totally rules out illumination. Or, the topography of the intervening surface may simply cast its shadow on the test element. The probability that a point on a surface is illuminated depends on the angle of incidence as well as the height and slope of the surface point.

In its application to scattering theory, the appropriate shadowing formalism[6] was found to depend on both the angles of incidence and observation. The application to transmission theory will be bistatic as well. The shadowing theory for transmission differs substantially from that for scattering, however. Thus, a surface high point is more likely to be illuminated by the incident ray than a surface low point, but the surface high point is less likely to be visible to an arbitrarily oriented observer below the surface than the corresponding low point (Fig. 1).

We consider the incident ray $\vec{k}_0$, located in the x-z plane, and the arbitrary observer direction $\vec{k}_{1T}$, defined as in Fig. 2. From Sancer,[7] the probability that surface point 1 is illuminated from $\vec{k}_0$, given its height and slope, is

$$S(\theta_0|z_1,s_{1x},s_{1y}) = u(-\hat{k}_0\cdot\hat{n}_1)\left[\frac{1}{2}\left(1+\text{erf}\,\frac{z_1}{(2\sigma^2)^{\frac{1}{2}}}\right)\right]^{2B_0} , \tag{6.1}$$

where

$$B_0 = (4(\pi)^{\frac{1}{2}}V_0)^{-1}\left[\exp(-V_0^2) - (\pi)^{\frac{1}{2}}V_0(1-\text{erf}\ V_0)\right] , \tag{6.2}$$

$$V_0 = \frac{\cot\theta_0}{\left[2\langle\zeta_x^2\rangle\right]^{\frac{1}{2}}} . \tag{6.3}$$

Again, the quantity u in Eq. (6.1) is a step function, which is unity if the argument is positive and which is zero if the argument is negative. The quantity B_0 varies from zero to infinity as θ_0 varies between normal and grazing incidence. Also, the height dependent factor in Eq. (6.1) varies from zero to unity (for

nonnormal incidence) as z_1 varies from minus to plus infinity. This is the desired behavior for the incident direction, but for the transmitted (observed) direction we want the inverse variation; i.e., as z_1 varies from plus to minus infinity, the probability of illumination should increase from zero to unity. By use of the same analysis applied to the derivation of Eq. (6.1), we conclude that

$$S(\theta_{1T},\phi_{1T}|z_1,s_{1x},s_{1y}) = u(-\hat{k}_{1T}\cdot\hat{n}_1)\left[\frac{1}{2}\left(1-\text{erf}\,\frac{z_1}{(2\sigma^2)^{\frac{1}{2}}}\right)\right]^{2B_{1T}} , \quad (6.4)$$

and B_{1T} follows from B_o by replacement of V_o by V_{1T}, where

$$V_{1T} = \frac{\cot\theta_{1T}}{\left[2\left(\langle\zeta_x^2\rangle\cos^2\phi_{1T} + \langle\zeta_y^2\rangle\sin^2\phi_{1T}\right)\right]^{\frac{1}{2}}} . \quad (6.5)$$

The desired quantity is the bistatic probability of illumination. Fortunately, because $\vec{k}_o$ and $\vec{k}_{1T}$ are associated with different quadrants of space, we can assume statistical independence:

$$S(\vec{k}_o,\vec{k}_{1T}|z_1,s_{1x},s_{1y}) = S(\theta_o|z_1,s_{1x},s_{1y})S(\theta_{1T},\phi_{1T}|z_1,s_{1x},s_{1y})$$

$$= u(-\hat{k}_o\cdot\hat{n}_1)u(-\hat{k}_{1T}\cdot\hat{n}_1)\left[\frac{1}{2}\left(1+\text{erf}\,\frac{z_1}{(2\sigma^2)^{\frac{1}{2}}}\right)\right]^{2B_o}\left[\frac{1}{2}\left(1-\text{erf}\,\frac{z_1}{(2\sigma^2)^{\frac{1}{2}}}\right)\right]^{2B_{1T}} . \quad (6.6)$$

Equation (6.6) provides more information than we need, as the transmission coefficient formalism of Section V requires only the probability of illunination, conditional on the slope value. Therefore, we simply average Eq. (6.6) over heights. For a normally distributed surface, the expression is

$$S(\vec{k}_o,\vec{k}_{1T}|s_{1x},s_{1y}) = \int_{-\infty}^{\infty} dz_1 P(z_1) S(\vec{k}_o,\vec{k}_{1T}|z_1,s_{1x},s_{1y}) . \quad (6.7)$$

By means of the substitution

$$y = \frac{1}{2}\left(1+\text{erf}\,\frac{z}{(2\sigma^2)^{\frac{1}{2}}}\right)$$

and the property

$$d/dz\ \text{erf}\left[\frac{z}{(2\sigma^2)^{\frac{1}{2}}}\right] = 2\ \overset{!}{P}(z) \quad ,$$

we have

$$S(\hat{k}_0,\hat{k}_{1T}|s_{1x},s_{1y}) = u(-\hat{k}_0\cdot\hat{n}_1)u(-\hat{k}_{1T}\cdot\hat{n}_1)\int_0^1 dy\ y^{2B_0}(1-y)^{2B_{1T}} \tag{6.8}$$

$$= \frac{u(-\hat{k}_0\cdot\hat{n}_1)u(-\hat{k}_{1T}\cdot\hat{n}_1)}{[1+2(B_0+B_{1T})]}\ \frac{\Gamma(1+2B_0)\Gamma(1+2B_{1T})}{\Gamma[1+2(B_0+B_{1T})]} \tag{6.9}$$

The quantities Γ are Gamma functions,[7] and the integral in Eq. (6.8) is, of course, the standard representation for the Beta function ($B_0,B_{1T} \geqslant 0$).

It needs to be emphasized here that Eq. (6.9) is valid for any arbitrary $\vec{k}_{1T}$. In the application to the transmission problem, however, $\vec{k}_{1T}$ is not an independent vector, but it is related to $\hat{k}_0$ and the surface normal $\hat{n}_1$ by Eq. (C.4); i.e.,

$$n(-\hat{k}_{1T}\cdot\hat{n}_1) = (-\hat{k}_0\cdot\hat{n}_1) + \{[n^2-1+\cos^2\alpha_1]^{\frac{1}{2}}-\cos\alpha_1\} \quad . \tag{6.10}$$

It is easily seen from this relation that if $(-\hat{k}_0\cdot\hat{n}_1) > 0$, then $(-\hat{k}_{1T}\cdot\hat{n}_1) > 0$ as well. Therefore, for the transmission problem, we can set $u(-\hat{k}_{1T}\cdot\hat{n}_1)$ equal to unity without loss of generality. In addition, we have already incorporated the step function $u(-\hat{k}_0\cdot\hat{n}_1)$ in the transmission theory of Section V in the context of the change of variables from (s_{1x},s_{1y}) to (θ_{1T},ϕ_{1T}). Thus, the slope coordinates (s^0_{1x},s^0_{1y}) in Eq. (5.12) can be assumed to be physically allowed. Therefore, we now convert

Eq. (6.9) to the notation of Section V in order to obtain

$$P[\varepsilon_1(\vec{k}_0,-\vec{k}) = 1|s^0_{1x},s^0_{1y}] \equiv S(\vec{k}_0,\vec{k}|s^0_{1x},s^0_{1y})$$

$$= \frac{1}{[1+2(B_0+B)]} \frac{\Gamma(1+2B_0)\Gamma(1+2B)}{\Gamma[1+2(B_0+B)]} \quad . \tag{6.11}$$

VII. REFLECTION FOLLOWED BY TRANSMISSION

Radiation incident on a rough surface will undergo multiple scatter due to that roughness. At each point of contact between ray and surface, there will be a contribution to the transmission of energy across the boundary. The coefficient $\langle t_1\rangle$ of Section V accounts only for the transmission into the medium at the initial point of contact [the ray $\vec{k}_{1T}$ in Fig. (3)]. In this section, we will write down the coefficient $\langle t_2\rangle$ appropriate to transmission at the second point of contact [the ray $\vec{k}_{2T}$ in Fig. (3)]. Only multiple-scatter ray diagrams of order two and higher will contribute to this term. A good estimate of the importance of this process can be obtained from Section III where a numerical evaluation of the quantities $T_1(\theta_0)$ and $T_2(\theta_0)$ has been carried through. From Tables I and II, we see that the ratio (T_2/T_1) has a broad maximum of roughly 3% at the large angles of incidence.

The partial-surface representation of $t_2(\theta,\theta_0)$ is given by Eq. (4.4). Because of the cumbersome mathematics involved, we will restrict the details to the one dimensional rough surface, $\zeta(x,y) = \zeta(x)$. Also, the averaging process is greatly simplified if we assume points 1 and 2 to be statistically independent. This appears to be an excellent approximation because the slopes at points 1 and 2 will usually be appreciably different in value, the characteristic of a separation distance larger than the correlation length. By use of techniques similar to those employed in Section V, we have

$$\left\langle t_2(\theta,\theta_o)\right\rangle = (\sec\theta_o)\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 \; P[s_1,s_2,I_1(\theta_o),I_{12}(\theta_1),I_2(\theta_{2T})]$$

$$\times \; u(\pi/2-|\theta_{2T}|) f_2(s_1,s_2)$$

$$\times \; \exp\{-K[\ell+\ell_2(\theta_1)]\sec\theta\}$$

$$\times \; \delta[\theta-\theta_{2T}(\theta_o,s_1,s_2)] \quad , \tag{7.1}$$

where

$$f_2(s_1,s_2) = (1+s_1^2)^{\frac{1}{2}}(\cos\alpha_1)r_1[\cos\alpha_1(s_1)]\{1-r_2[\cos\alpha_2(s_1,s_2)]\} \quad , \tag{7.2}$$

$$\ell_2(\theta_1) = \int_{-\infty}^{\infty} dz_2 \; z_2 P[z_2|I_1(\theta_o), \; I_{12}(\theta_1),I_2(\theta)] \quad . \tag{7.3}$$

The quantity P is a joint probability density on slope and illumination for each of the two scatter points. The symbol $I_{12}(\theta_1)$ denotes the likelihood that the intermediate ray $\vec{k}_1$ actually intersects the surface again at point 2; it must be included as a fifth random variable in any consistent theory for double scatter.

As the argument of the delta function depends on θ_{2T} rather than s_2, it is clearly advisable to change variables from (s_1,s_2) to (s_1,θ_{2T}). The equation of transformation can be deduced from Appendix C:

$$s_2 = -\frac{(n\sin\theta_{2T}+\sin\theta_1)}{(n\cos\theta_{2T}+\cos\theta_1)} \quad , \tag{7.4}$$

where

$$\theta_1 = -\theta_o-2\tan^{-1}s_1 \quad , \tag{7.5}$$

or, inversely,

$$s_1 = -\tan\left(\frac{\theta_o+\theta_1}{2}\right) \quad . \tag{7.6}$$

The Jacobian of the transformation is

$$J(s_1) = \frac{n[n+\cos(\theta_{2T}-\theta_1)]}{(\cos\theta_1+n\cos\theta_{2T})^2} \quad , \tag{7.7}$$

a function of θ_{2T} and, implicitly, of s_1. As was the case with the direct transmission coefficient (Section V), the range of integration for θ_{2T} is restricted by the illumination condition, $(-\hat{k}_1\cdot\hat{n}_2) > 0$, and by the condition $(\hat{n}_2\cdot\hat{z}) > 0$, the latter a consequence of single-valued normally distributed surfaces. An additional complication arises because some values of s_1 lead only to values of $|\theta_{2T}|$ larger than $\pi/2$ (e.g., the vector $\vec{k}'_{2T}$ in Fig. 3). The following expression for $\langle t_2 \rangle$ accounts for the change of variables, the resulting integration over θ_{2T}, and the imposition of the preceding restrictions:

$$\begin{aligned}
\left\langle t_2(\theta,\theta_o)\right\rangle = (\sec\theta_o)\Bigg[& \int\limits_{\sin^{-1}(\frac{1}{n}) < \theta_1 < \pi/2} ds_1 \; J(s_1)u(-\chi_1^L-\theta)F(s_1,s_2^o) \\
& + \int\limits_{\theta_1>\pi/2} ds_1 \; J(s_1)u(\theta+\chi_1^U)u(-\chi_1^L-\theta)F(s_1,s_2^o) \\
& + \int\limits_{-\pi/2<\theta_1<-\sin^{-1}(\frac{1}{n})} ds_1 \; J(s_1)u(\theta-\chi_1^L)F(s_1,s_2^o) \\
& + \int\limits_{\theta_1<-\pi/2} ds_1 \; J(s_1)u(\theta-\chi_1^L)u(\chi_1^U-\theta)F(s_1,s_2^o)\Bigg] \quad ,
\end{aligned} \tag{7.8}$$

where the integration ranges for the slope integrals are given in terms of equivalent ranges for the angle θ_1 (Eq. 7.6). The quantities χ_1^L, χ_1^U in the step-function arguments are defined as

$$\chi_1^U = \pi/2 + \sin^{-1}\left(\frac{\cos\theta_1}{n}\right) \quad , \tag{7.9}$$

$$\chi_1^L = (\pi-|\theta_1|) - \left[\pi/2-\sin^{-1}(\tfrac{1}{n})\right] \quad . \tag{7.10}$$

The quantity $F(s_1,s_2^0)$ is

$$F(s_1,s_2^0) = P[s_1,s_2^0,I_1(\theta_0), I_{12}(\theta_1),I_2(\theta)]f_2(s_1,s_2^0)$$
$$\times \exp\{-K[\ell+\ell_2(\theta_1)]\sec\theta\} \tag{7.11}$$

where

$$s_2^0 = -\frac{(\sin\theta_1+n\sin\theta)}{(\cos\theta_1+n\cos\theta)} \quad , \tag{7.12}$$

and f_2 is given by Eq. (7.2), evaluated at $s_2 = s_2^0$. The arguments of the reflection coefficients can be expressed most usefully as

$$\cos\alpha_1(s_1) = (1+s_1^2)^{-\frac{1}{2}}(\cos\theta_0 - s_1\sin\theta_0) \quad , \tag{7.13}$$

$$\cos\alpha_2(s_1,s_2^0) = -\left[1+(s_2^0)^2\right]^{-\frac{1}{2}}(\cos\theta_1-s_2^0\sin\theta_1) \quad . \tag{7.14}$$

The fifth-order probability density can be written as the product of five probability functions:

$$P[s_1,s_2^0,I_1(\theta_0), I_{12}(\theta_1),I_2(\theta)] = P(s_1)P_1P_2P_3P_4 \quad , \tag{7.15}$$

where

$$P_1 = P[I_1(\theta_0)|s_1] \quad , \tag{7.16}$$

$$P_2 = P[I_{12}(\theta_1)|s_1,I_1(\theta_0)] \quad , \tag{7.17}$$

$$P_3 = P[s_2^0 | s_1, I_1(\theta_0), I_{12}(\theta_1)] \quad , \tag{7.18}$$

$$P_4 = P[I_2(\theta) | s_1, s_2^0, I_1(\theta_0), I_{12}(\theta_1)] \quad . \tag{7.19}$$

The probability density $P(s_1)$ is the probability density of slopes for a normally distributed surface. The mathematical relations for P_1, P_2, and P_3 can be found in the literature.[4] The remaining P_4 asks for the probability that an observer in the medium, oriented at angle θ, receives the refracted ray, given the values of slope at points 1 and 2 and given the facts of illumination specified by $I_1(\theta_0)$ and $I_{12}(\theta_1)$. The analytical results for P_4 are

$$P_4 = \frac{1}{(1+2B)} \quad , \qquad |\theta_1| < \pi/2$$

$$= \frac{\left(\dfrac{1}{1+2B}\right) - \left\{\dfrac{1}{[1+2(B_1+B)]}\,\dfrac{\Gamma(1+2B_1)\Gamma(1+2B)}{\Gamma[1+2\,(B_1+B)]}\right\}}{\left(\dfrac{2B_1}{1+2B_1}\right)} \,, \qquad |\theta_1| > \pi/2 \;, \tag{7.20}$$

where the B's have the functional dependence given by Eq. (5.27) and V, V_1 are given by

$$V = \frac{\cot\theta}{\left[2\langle\zeta_x^2\rangle\right]^{\frac{1}{2}}} \quad , \tag{7.21}$$

$$V_1 = \frac{|\cot(\pi-\theta_1)|}{\left[2\langle\zeta_x^2\rangle\right]^{\frac{1}{2}}} \quad . \tag{7.22}$$

The only remaining quantity to be specified is $\ell_2(\theta_1)$, the average illuminated height of point 2. The results are

$$\ell_2(\theta_1) = \frac{-B}{(1+B)} (\pi/2)^{\frac{1}{2}} \sigma \quad , \quad |\theta_1| < \pi/2$$

$$= -(\pi/2)^{\frac{1}{2}} \sigma \frac{\left[\left(\frac{B}{1+B}\right) \Gamma(2+2B_1+2B) + \frac{(B_1-B)}{(1+B_1+B)} \Gamma(1+2B_1)\Gamma(2+2B)\right]}{\left[\Gamma(2+2B_1+2B) - \Gamma(1+2B_1)\Gamma(2+2B)\right]} ,$$

$$|\theta_1| > \pi/2 \quad . \tag{7.23}$$

VIII. TRANSMISSION FOLLOWED BY REFLECTION

The partial transmission coefficients t_1 and t_2 describe a transmitted ray which travels directly to a hypothetical observer; i.e., the transmitted ray does not intersect the surface again. However, Fig. 4 demonstrates a higher-order process with transmission at point 1 followed by a surface reflection at point 2 (the ray $\vec{k}_{2R}$). In this section we derive the average transmission coefficient $\langle t_{1R} \rangle$ associated with this process. This effect is included only for completeness as its contribution to the transmitted radiation pattern is expected to be quite small. This follows because, as Fig. 4 implies, the angle $\theta_{1T} (\theta_{1T} < \pi/2)$ must be quite large if the ray is to have much chance of intersecting the surface again. But it is easy to show that a large value of θ_{1T} requires a combination of large angle of incidence and an element of surface with steep slope. Thus, $\langle t_{1R} \rangle$ can contribute significantly only for near-grazing incidence and the very roughest of ocean surfaces. We restrict the details in the following to the one dimensional rough surface, $\zeta(x,y) = \zeta(x)$.

The partial-surface representation for $t_{1R}(\theta,\theta_0)$ is given by Eq. (4.3). The assumption that points 1 and 2 are statistically independent again appears to be an excellent one, for the slope values at the two points must be very different. This implies a separation distance greater than the correlation length. The average value of t_{1R} follows by the techniques of Section V:

$$\left\langle t_{1R}(\theta,\theta_o)\right\rangle = (\sec\theta_o)\int_{-\infty}^{\infty}ds_1\int_{-\infty}^{\infty}ds_2\; P[s_1,s_2,I_1(\theta_o),I_{12}(\theta_{1T}),I_2(\theta_{2R})] \times$$

$$\times\; u(\pi/2-|\theta_{2R}|)f_{1R}(s_1,s_2)$$

$$\times\; \exp\{-K[\ell_3(\theta_{1T})-\ell_4(\theta_{1T})]\sec\theta_{1T}\}$$

$$\times\; \exp[-K(\ell+\ell_4)\sec\theta]$$

$$\times\; \delta[\theta-\theta_{2R}(\theta_o,s_1,s_2)] \quad , \tag{8.1}$$

where

$$f_{1R}(s_1,s_2) = (1+s_1^2)^{\frac{1}{2}}(\cos\alpha_1)[1-r_1(\cos\alpha_1)]r_2(\hat{k}_{1T}\cdot\hat{n}_2) \quad , \tag{8.2}$$

$$\ell_3(\theta_{1T}) = \int_{-\infty}^{\infty}dz_1\; z_1\; P[z_1|I_1(\theta_o),\; I_{12}(\theta_{1T})] \quad , \tag{8.3}$$

$$\ell_4(\theta_{1T}) = \int_{-\infty}^{\infty}dz_2\; z_2\; P[z_2|I_{12}(\theta_{1T}),\; I_2(\theta_{2R})] \quad , \tag{8.4}$$

and

$$\cos\alpha_1 = (1+s_1^2)^{-\frac{1}{2}}(\cos\theta_o - s_1\sin\theta_o) \quad , \tag{8.5}$$

$$\hat{k}_{1T}\cdot\hat{n}_2 = -(1+s_2^2)^{-\frac{1}{2}}(\cos\theta_{1T}-s_2\sin\theta_{1T}) \quad . \tag{8.6}$$

The quantity P is the joint probability density associated with this process.

We now change variables from (s_1,s_2) to $(\theta_{1T},\theta_{2R})$. The equations of transformation are (from Appendix C and Reference 4)

$$s_1 = -\frac{(\sin\theta_o - n\sin\theta_{1T})}{(\cos\theta_o - n\cos\theta_{1T})} , \qquad (8.7)$$

$$s_2 = \tan\left[\frac{(\pi-\theta_{1T}-\theta_{2R})}{2}\right] . \qquad (8.8)$$

The Jacobian of the transformation is

$$J(\theta_{1T},\theta_{2R}) = \frac{n}{2}\,\frac{\left[n-\cos(\theta_o-\theta_{1T})\right]}{(\cos\theta_o - n\cos\theta_{1T})^2}\,\sec^2\left[\frac{(\pi-\theta_{1T}-\theta_{2R})}{2}\right] . \qquad (8.9)$$

A little geometry shows that the allowed range of θ_{2R} is $-|\theta_{1T}| < \theta_{2R} < |\theta_{1T}|$. Thus, since $|\theta_{1T}|$ is always less than $\pi/2$, we have $|\theta_{2R}| < \pi/2$, and the step function in Eq. (8.1) is always unity. The range of θ_{1T} is determined by the now familiar conditions $(\hat{n}_1\cdot\hat{z}) > 0$ and $(-\hat{k}_o\cdot\hat{n}_1) > 0$. The result is $\chi^L < \theta_{1T} < \chi^U$, where

$$\chi^U = \pi/2 - \sin^{-1}\left(\frac{\cos\theta_o}{n}\right) , \qquad (8.10)$$

$$\chi^L = \theta_o - \left[\pi/2 - \sin^{-1}\left(\frac{1}{n}\right)\right] . \qquad (8.11)$$

By use of the transformation equations, the final form for $\langle t_{1R}\rangle$ can now be written down:

$$\begin{aligned}\langle t_{1R}(\varepsilon,\theta_o)\rangle = (\sec\theta_o)\int_{\chi^L}^{\chi^U} d\theta_{1T}\; & J(\theta_{1T},\theta)u(|\theta_{1T}|-|\theta|)f_{1R}[s_1(\theta_{1T}),s_2(\theta_{1T},\theta)] \\ & \times P[s_1(\theta_{1T}),s_2(\theta_{1T},\theta),I_1(\theta_o),I_{12}(\theta_{1T}),I_2(\theta)] \\ & \times \exp\{-K[\ell_3(\theta_{1T})-\ell_4]\sec\theta_{1T}\} \\ & \times \exp[-K(\ell+\ell_4)\sec\theta] , \end{aligned} \qquad (8.12)$$

where

$$s_2^o(\theta_{1T},\theta) = \tan\left[\frac{(\pi-\theta_{1T}-\theta)}{2}\right] . \qquad (8.13)$$

The probability density P can be written as the product of five probability functions:

$$P[s_1,s_2^0,I_1(\theta_o),I_{12}(\theta_{1T}),I_2(\theta)] = P(s_1)P_1P_2'P_3'P_4' \quad , \tag{8.14}$$

where

$$P_1 = P[I_1(\theta_o)|s_1] \quad , \tag{7.16}$$

$$P_2' = P[I_{12}(\theta_{1T})|s_1,I_1(\theta_o)] \quad , \tag{8.15}$$

$$P_3' = P[s_2|s_1,I_1(\theta_o),I_{12}(\theta_{1T})] \quad , \tag{8.16}$$

$$P_4' = P[I_2(\theta)|s_1,s_2,I_1(\theta_o),I_{12}(\theta_{1T})] \quad . \tag{8.17}$$

The mathematical relations for these probabilities are

$$P_1 = \left(\frac{1}{1+2B_o}\right) u(\cot\theta_o-s_1) \quad , \tag{8.18}$$

$$P_2' = 1 - \frac{S(\theta_o,\theta_{1T}|s_1)}{P_1} \quad , \tag{8.19}$$

where the bistatic shadowing probability is given by Eq. (6.9);

$$\begin{aligned} P_3' &= \left(\frac{2}{1-\mathrm{erf}\ V_{1T}}\right) u(s_2-\cot\theta_{1T})P(s_2) \ , \qquad 0<\theta_{1T}<\pi/2 \\ &= \left(\frac{2}{1-\mathrm{erf}\ V_{1T}}\right) u(\cot\theta_{1T}-s_2)P(s_2) \ , \qquad \theta_{1T}<0 \quad , \end{aligned} \tag{8.20}$$

$$P_4' = \left(\frac{1}{1+2B}\right) \quad . \tag{8.21}$$

The quantities B and B_0 are given by Eqs. (5.27) and (6.2), while

$$V_{1T} = \frac{|\cot\theta_{1T}|}{\left[2\left\langle\zeta_x^2\right\rangle\right]^{\frac{1}{2}}} \quad . \tag{8.22}$$

The remaining quantities which must be specified are ℓ_3 and ℓ_4, the average illuminated heights of points 1 and 2 for this process. The results are

$$\ell_3(\theta_{1T}) = (\pi/2)^{\frac{1}{2}} \sigma \frac{\left[\frac{B_o}{(1+B_o)}\Gamma(2+2B_o+2B_{1T}) - \frac{(B_o-B_{1T})}{(1+B_o+B_{1T})}\Gamma(1+2B_{1T})\Gamma(2+2B_o)\right]}{\left[\Gamma(2+2B_o+2B_{1T}) - \Gamma(1+2B_{1T})\Gamma(2+2B_o)\right]} , \quad (8.23)$$

$$\ell_4(\theta_{1T}) = \ell_4 = \frac{-B}{(1+B)} (\pi/2)^{\frac{1}{2}} \sigma \quad . \quad (8.24)$$

IX. RECIPROCITY, AND TRANSMISSION INTO A LESS REFRACTIVE MEDIUM

In the previous sections, our investigations were concerned with the transmission of light from a medium of index unity into a medium of refractive index $n > 1$. We are, of course, interested in the reverse problem of transmission into the less refractive medium. Thus, for the sake of argument, we could consider a source located at depth $z = -\ell$, with plane radiation emitted into direction $\vec{k}_s$ towards the surface. The strength of the radiation refracted into the air and traveling in direction $\vec{k}'$ away from the surface is then of interest. Therefore, we want to derive $\left\langle t^{ij}(\vec{k};\vec{k}_s)\right\rangle_{n\to 1}$.

It is expected that the solution of the $(n\to 1)$ transmission problem will follow from the already derived $(1\to n)$ transmission coefficients. This follows because the transmission coefficient must have a reciprocity property associated with it which relates to the interchange of source and observer. Reciprocity is a general statement of the dynamical reversibility of the system, so its mathematical expression must be independent of the shape of the surface. Therefore, we will derive the reciprocity theorem for a plane surface and then simply infer its form for the general rough surface. We will also assume n to be real, which is quite valid for the optical transmission problem.

Consider plane-wave radiation incident on a flat surface with normal in the direction of the z axis. We choose this orientation because the transmission and scattering properties of any stationary random rough surface must reduce in the

long-wavelength limit to the transmission and scattering results for this plane surface. We will initially restrict the incident polarization to be either horizontal or vertical relative to the local plane of incidence; this will provide a reciprocity relation for one dimensional surfaces. The transmission coefficients of interest can be written down by inspection for the plane surface:

$$t_p(\vec{k},\vec{k}_o) = [1-r(\cos\theta_o)]\delta\left[\theta-\sin^{-1}\left(\frac{\sin\theta_o}{n}\right)\right] \quad , \tag{9.1}$$

$$t_p(-\vec{k}_o,-\vec{k}) = [1-\bar{r}(\cos\theta)]\delta[\theta_o-\sin^{-1}(n\sin\theta)] \quad , \tag{9.2}$$

where the reflectivities are those appropriate to either horizontal or vertical polarization. For example, Eq. (9.1) states that the transmission coefficient is zero unless the angle of observation coincides with the angle of refraction for a plane surface as calculated by Snell's law. The bar on the reflectivity in Eq. (9.2) is a reminder that for the $(1\rightarrow n)$ process, we have, e.g.,

$$R_H(\cos\theta_o) = \frac{\cos\theta_o - \left[n^2-\sin^2\theta_o\right]^{\frac{1}{2}}}{\cos\theta_o + \left[n^2-\sin^2\theta_o\right]^{\frac{1}{2}}} \quad , \tag{9.3}$$

but for the $(n\rightarrow 1)$ process,

$$\bar{R}_H(\cos\theta) = \frac{\cos\theta - \left[\frac{1}{n^2} - \sin^2\theta\right]^{\frac{1}{2}}}{\cos\theta + \left[\frac{1}{n^2} - \sin^2\theta\right]^{\frac{1}{2}}} \quad . \tag{9.4}$$

We now use the relation

$$\delta[f(\theta)] = \frac{\delta(\theta-\theta^*)}{|f'(\theta^*)|} \quad , \tag{9.5}$$

where

$$f(\theta) = \theta_o-\sin^{-1}(n\sin\theta) \quad , \tag{9.6}$$

and θ^* is the solution to

$$f(\theta^*) = 0 \quad . \tag{9.7}$$

The root θ^* is easily seen to be

$$\theta^* = \sin^{-1}\left(\frac{\sin\theta_o}{n}\right) \quad , \tag{9.8}$$

and, since,

$$|f'(\theta)| = \frac{n\cos\theta}{(1-n^2\sin^2\theta)^{\frac{1}{2}}} \quad , \tag{9.9}$$

we have, by use of the property of the delta function,

$$t_p(-\vec{k}_o,-\vec{k}) = [1-\bar{r}(\cos\theta^*)]\delta(\theta-\theta^*)\,\frac{(1-n^2\sin^2\theta^*)^{\frac{1}{2}}}{n\cos\theta} \quad . \tag{9.10}$$

But it is easily shown that

$$\bar{r}(\cos\theta^*) = r(\cos\theta_o) \quad , \tag{9.11}$$

$$(1-n^2\sin^2\theta^*)^{\frac{1}{2}} = \cos\theta_o \quad , \tag{9.12}$$

so

$$t_p(-\vec{k}_o,-\vec{k}) = \frac{\cos\theta_o}{n\cos\theta}\,[1-r(\cos\theta_o)]\delta(\theta-\theta^*)$$

$$t_p(-\vec{k}_o,-\vec{k}) = \left(\frac{\cos\theta_o}{n\cos\theta}\right) t_p(\vec{k},\vec{k}_o) \quad . \tag{9.13}$$

This is not the most general form of the reciprocity relation, for the t's are defined "per unit sector of angle," and angular widths are not invariant in the transmission process. In order to extend Eq. (9.13), we again illuminate the flat plane with radiation of constant intensity but now with a narrow sector $d\theta_o$

of illumination angles about the angle θ_o. From Snell's law, $\sin\theta_o = n\sin\theta$, the angular width of the corresponding refracted radiation is

$$d\theta = \frac{\cos\theta_o}{n\cos\theta}\, d\theta_o \quad . \tag{9.14}$$

The substitution of this result into Eq. (9.13) yields

$$d\theta\ t_p(\vec{k},\vec{k}_o) = d\theta_o\ t_p(-\vec{k}_o,-\vec{k}) \quad . \tag{9.15}$$

The left-hand side of Eq. (9.15) is the fraction of the incident energy which is transmitted. This is evidently equal to the fraction transmitted in the reciprocal process, i.e., with the source in the medium and radiation of width $d\theta$ about direction $-\vec{k}$ incident at the interface.

The value of Eq. (9.15) is that we can generalize to "two dimensional" results by simply replacing angular sectors by solid angles and by introducing polarization notation in the usual way; i.e.,

$$(d\phi d\theta \sin\theta) t_p^{ji}(\vec{k},\vec{k}_o) = (d\phi_o d\theta_o \sin\theta_o)\ t_p^{ij}(-\vec{k}_o,-\vec{k}) \quad . \tag{9.16}$$

Since the incident ray, the normal, and the refracted ray all lie in the same plane, we have $d\phi = d\phi_o$. Then, by use of Snell's law and Eq. (9.14),

$$t_p^{ij}(-\vec{k}_o,-\vec{k}) = \frac{\cos\theta_o}{n^2\cos\theta}\ t_p^{ji}(\vec{k},\vec{k}_o) \quad . \tag{9.17}$$

Because of the general nature of the reciprocity principle, we expect that the coefficients for the general random rough surface satisfy

$$\left\langle t^{ij}(-\vec{k}_o,-\vec{k})\right\rangle = \left(\frac{\cos\theta_o}{n^2\cos\theta}\right)\left\langle t^{ji}(\vec{k},\vec{k}_o)\right\rangle \quad . \tag{9.18}$$

Finally, we replace $-\vec{k}\to\vec{k}_s$, $-\vec{k}_o\to\vec{k}'$ to obtain

$$\left\langle t^{ij}(\vec{k}',\vec{k}_s)\right\rangle = \left(\frac{\cos\theta'}{n^2\cos\theta_s}\right)\left\langle t^{ji}(-\vec{k}_s,-\vec{k}')\right\rangle \quad . \tag{9.19}$$

We can now substitute the results of Sections V, VII, and VIII into the right-hand side of Eq. (9.19) to obtain the corresponding partial transmission coefficients for refraction from water into air.

X. INTENSITY RELATION FOR UPWELLING RADIATION

Upwelling radiation from beneath the surface is an important component of sea color. Such radiation originates from the backscattering of transmitted sunlight by both molecular and particulate matter or, in shallow water, by the sea bottom. Some of this scattered radiation is reflected back into the medium by the sea surface, but a certain fraction will be transmitted into the atmosphere. The intensity reaching an observer above the surface in direction $\vec{k}$ will depend on the original intensity, the transmission coefficients for the (rough) air-water and water-air interfaces, and on the scattering coefficient defined as

$\gamma^{ij}(\vec{k}_2,\vec{k}_1)$ = fraction of incident ($\vec{k}_1$) radiation, with polarization i, which is scattered into direction $\vec{k}_2$, in polarization state j.

The expression for the intensity of the upwelling radiation, in polarization state n, is clearly

$$I^{in}(\vec{k}) = \frac{1}{\cos\theta} \sum_{j=1}^{2} \sum_{m=1}^{2} \int d\Omega_s \int d\Omega_t \int d\Omega_o \, t^{mn}(\vec{k},\vec{k}_s)\gamma^{jm}(\vec{k}_s,\vec{k}_t) \times t^{ij}(\vec{k}_t,\vec{k}_o)(\cos\theta_o)I_o^i(\vec{k}_o) \tag{10.1}$$

where $I_c^i(\vec{k}_o)$ is the initial intensity, in polarization state i.

Strictly speaking, the transmission coefficients for the physical-ocean medium will differ from those derived in the previous sections on the basis of an idealized semi-infinite homogeneous medium. The physical medium has a fluctuating index of refraction which makes the description of propagation much more complicated than the exponential absorption model adopted here. Nevertheless, the t^{ij} of this report should be sufficiently accurate for clear waters. Also, we will assume that the distance to the bottom is large compared with the absorption length

so that any radiation scattered off the bottom and subsequently scattered back into the medium by the surface will be of negligible intensity as compared to the direct transmission component.

The simplest application of Eq. (10.1) is to a homogeneous (but absorbing) medium with a plane bottom (of zero slope). Then, the scattering coefficient is

$$\gamma^{jm}(\vec{k}_s,\vec{k}_t) = r_j(\hat{k}_t\cdot\hat{z})\delta^{jm}\delta(\hat{k}_s-\hat{k}_s^*) \quad , \tag{10.2}$$

where

$$\hat{k}_s^* = \hat{k}_t-2(\hat{k}_t\cdot\hat{z})\hat{z} \quad . \tag{10.3}$$

Here, r_j is the reflectivity of the bottom for radiation in polarization state j. We also assume a plane wave for the incident radiation, so

$$I_o^i(\vec{k}_o) = I_o^i\delta(\hat{k}_o-\hat{k}_o') \quad . \tag{10.4}$$

Then, the substitution of Eqs. (10.2) and (10.4) into Eq. (10.1) yields

$$I^n(\vec{k}) = \left(\frac{\cos\theta_o'}{\cos\theta}\right)I_o^i \sum_{j=1}^{2} \int d\Omega_t \; t^{jn}(\vec{k},\vec{k}_s^*)r_j(\hat{k}_t\cdot\hat{z})t^{ij}(\vec{k}_t,\hat{k}_o') \quad . \tag{10.5}$$

All quantities in this equation are known, with the transmission coefficients available from Sections V and IX.

XI. CONCLUSIONS

A ray theory which describes the transmission of light across a random rough boundary has been formulated. The approach uses a complete geometrical-optics treatment in that multiple-scatter and shadowing effects are retained. The main results of the analysis are as follows:

1) A theory of shadowing for the transmission problem has been derived; i.e., the probability that a surface point is visible to a source above the surface as well as to an observer below the surface (or vice versa) is now available in closed form.

2) The transmission coefficient, which determines the angular profile of transmitted energy, is expressed as an expansion in terms of the number of ray intersections with the surface. All terms in the expansion are positive, so the retention of, say, the first m terms yields a lower bound to the true value of the transmission coefficient. The leading term, the direct transmission coefficient, is corrected for shadowing but contains no multiple-scatter information. As expected, it is proportional to the probability density of that slope which connects the incident and observer directions by Snell's law. The direct transmission coefficient is derived for the general two dimensional random rough surface, and it therefore contains polarization information. Expressions for both air-to-water and water-to-air transmission are available, the latter by a reciprocity argument.

3) Higher-order transmission terms were formulated only for the mathematically tractable one dimensional rough surface (cylindrical symmetry). Thus, a ray of sunlight may reflect from one point on the ocean surface only to intersect the surface again at a second point. The angular profile due to transmission of energy at the second point has been derived; the total fraction of incident energy associated with this double-scatter process is about 3% of the directly transmitted energy for likely ocean roughnesses. The corresponding process in which a transmitted ray of sunlight is reflected by the under-sea surface was also formulated. It is expected to contribute significantly to the below-surface radiation profile only for near-grazing incidence and the very roughest of ocean surfaces.

APPENDIX A

We consider the general case of a medium with complex index of refraction $\tilde{n}$, where

$$\tilde{n} = n + in_i \quad . \tag{A.1}$$

Snell' [illegible] has the familiar form for radiation incident on a plane surface at angle θ_0,

$$\sin\tilde{\theta}_T = \frac{\sin\theta_0}{\tilde{n}} \quad , \tag{A.2}$$

but since $\tilde{n}$ is complex, the quantity $\tilde{\theta}_T$ is also complex and no longer has the simple significance of the angle of refraction. The angle that is significant, θ_T, relates the direction of propagation of the surfaces of constant real phase with the normal to the boundary. From Born and Wolf,[8] we have

$$\sin\theta_T = \frac{\sin\theta_0}{D} \quad , \tag{A.3}$$

where

$$D = \left[\sin^2\theta_0 + n^2q^2\left(\cos\gamma - \frac{n_i}{n}\sin\gamma\right)^2\right]^{\frac{1}{2}} \tag{A.4}$$

and q and γ are real numbers defined by

$$\cos\tilde{\theta}_T = q\,\exp(i\gamma) \quad .$$

For optical radiation in the blue-green band, the attenuation length for clear water ranges up to about 30 meters, so at the peak $K = kn_i \simeq 3\times10^{-4}\mathrm{cm}^{-1}$, and n_i is a few times 10^{-9}. When $n_i/n \ll 1$, the quantity D reduces to

$$D \simeq n\left(1 - \frac{n_i^2}{n^4}\sin^2\theta_0\right) \quad . \tag{A.5}$$

Since the correction term is only of the order of 10^{-18} for the optical (blue-green) transmission problem (n = 1.33), Eq. (A.3) takes the form

$$\sin\theta_T \simeq \frac{\sin\theta_0}{n} \quad . \tag{A.6}$$

APPENDIX B

We are concerned here with refraction across an arbitrarily oriented plane surface (Fig. 5). First, let us consider the incident plane wave to be horizontally polarized, i.e., the electric field vector is orthogonal to the $(\hat{k}_o-\hat{z})$ plane of incidence:

$$\vec{E}_{inc} = \hat{y}\exp(i\vec{k}_o\cdot\vec{r}) \; ; \qquad \hat{k}_o = -\sin\theta_o\hat{x} - \cos\theta_o\hat{z} \; . \tag{B.1}$$

We now transform to the local $(\hat{k}_o-\hat{n}_1)$ plane of incidence:

$$\hat{y} = a_1\hat{v}_1+b_1\hat{h}_1 \quad , \tag{B.2}$$

where

$$\hat{h}_1 = \frac{(\hat{k}_o\times\hat{n}_1)}{|\hat{k}_o\times\hat{n}_1|} \quad , \tag{B.3}$$

$$\hat{v}_1 = -\hat{k}_o\times\hat{h}_1 \quad , \tag{B.4}$$

and, after simplification,

$$a_1 = \frac{\hat{n}_1\cdot\hat{y}}{|\hat{k}_o\times\hat{n}_1|} \quad , \tag{B.5}$$

$$b_1 = \frac{\hat{n}_1\cdot\hat{v}_o}{|\hat{k}_o\times\hat{n}_1|} \; ; \qquad \hat{v}_o = -\cos\theta_o\hat{x} + \sin\theta_o\hat{z} \; . \tag{B.6}$$

The incident field now has the canonical form

$$\vec{E}_{inc} = a_1\hat{v}_1\, e^{i\vec{k}_o\cdot\vec{r}} + b_1\hat{h}_1\, e^{i\vec{k}_o\cdot\vec{r}} \quad , \tag{B.7}$$

so the refracted field is

$$\vec{E}^{H}_{refr} = a_1 t_V \hat{V}_{1T} e^{i\vec{k}_{1T}\cdot\vec{r}} + b_1 t_H \hat{h}_1 e^{i\vec{k}_{1T}\cdot\vec{r}} , \tag{B.8}$$

where

$$\hat{V}_{1T} = \hat{h}_1 \times \hat{k}_{1T} , \tag{B.9}$$

and the superscript on the refracted field indicates the initial polarization state. The quantities t_V and t_H are determined in the usual way by requiring continuity of the tangential electric and magnetic field components at the surface:

$$t_H = 1 + R_H(\cos\alpha_1) , \tag{B.10}$$

$$t_V = \frac{1+R_V(\cos\alpha_1)}{n} , \tag{B.11}$$

where, as usual, $\cos\alpha_1 = (-\hat{k}_o \cdot \hat{n}_1)$.

Equation (B.8) is not in useful form for $\hat{V}_{1T}$ and $\hat{h}_1$ are polarization vectors for the local plane, while we require the polarization vectors for the $(\hat{k}_{1T}-\hat{z})$ plane of observation:

$$\vec{p}_H = \frac{(\hat{k}_{1T} \times \hat{z})}{|\hat{k}_{1T} \times \hat{z}|} , \tag{B.12}$$

$$\vec{p}_V = -\hat{k}_{1T} \times \vec{p}_H . \tag{B.13}$$

Thus, $\hat{V}_{1T}$ and $\hat{h}_1$ can be expanded in terms of these unit vectors:

$$\hat{V}_{1T} = c_1\vec{p}_V - c_2\vec{p}_H ,$$

$$\hat{h}_1 = c_1\vec{p}_H + c_2\vec{p}_V , \tag{B.14}$$

with

$$c_1 = \hat{h}_1 \cdot \vec{p}_H \quad , \tag{B.15}$$

$$c_2 = \hat{h}_1 \cdot \vec{p}_V \quad . \tag{B.16}$$

Equation (B.8) now takes the form

$$\vec{E}^H_{refr} = (a_1 c_1 t_V + b_1 c_2 t_H)\, \vec{p}_V\, e^{i\vec{k}_{1T}\cdot\vec{r}} + (b_1 c_1 t_H - a_1 c_2 t_V)\, \vec{p}_H\, e^{i\vec{k}_{1T}\cdot\vec{r}} \quad . \tag{B.17}$$

The time averaged Poynting vector, $\vec{\vec{N}} = \frac{1}{2}\,\mathrm{Re}(\vec{E} \times \vec{H}^*)$, can now be used to find the fraction of energy incident on the plane surface, in polarization state H, which is refracted into final polarization state H or V. We designate these fractions as τ^{HH} and τ^{HV} :

$$\tau^{HH} \equiv \frac{|\vec{\vec{N}}^{H\to H}_{refr}|}{|\vec{\vec{N}}^{H}_{inc}|} \left(\frac{\cos\alpha_1'}{\cos\alpha_1}\right) \tag{B.18}$$

$$= n|b_1 c_1 t_H - a_1 c_2 t_V|^2 \left(\frac{\cos\alpha_1'}{\cos\alpha_1}\right) \quad , \tag{B.19}$$

and, similarly,

$$\tau^{HV} = n|a_1 c_1 t_V + b_1 c_2 t_H|^2 \left(\frac{\cos\alpha_1'}{\cos\alpha_1}\right) \quad , \tag{B.20}$$

where, by Snell's law,

$$n\cos\alpha_1' = \left[(n^2-1) + \cos^2\alpha_1\right]^{\frac{1}{2}} \quad .$$

The alternate quantities τ^{VH} and τ^{VV}, corresponding to vertically polarized incident radiation, follow from Eqs. (B.19) and (B.20) by the replacement $a_1 \to b_1$, $b_1 \to -a_1$:

$$\tau^{VH} = n|a_1c_1t_H + b_1c_2t_V|^2\left(\frac{\cos\alpha_1'}{\cos\alpha_1}\right) \quad , \qquad (B.21)$$

$$\tau^{VV} = n|a_1c_2t_H - b_1c_1t_V|^2\left(\frac{\cos\alpha_1'}{\cos\alpha_1}\right) \quad . \qquad (B.22)$$

APPENDIX C

Snell's law relates the angle of refraction to the angle of incidence on a plane surface, but the angles are defined relative to the surface normal. For an irregular boundary, we must apply Snell's law to each of the locally flat and arbitrarily oriented elements comprising the surface. Then it is convenient to talk about the angles of incidence and refraction defined relative to a single coordinate system.

For isotropic media, the incident ray, the refracted ray, and the normal to the surface all lie in the same plane (Fig. 6). We want to derive the vector relation between the direction of the refracted ray and the directions of the incident ray and surface normal. One triplet of unit vectors is $-\hat{k}_0, \frac{(\hat{k}_0 \times \hat{k}_{1T})}{|\hat{k}_0 \times \hat{k}_{1T}|}$, and $\hat{V}$, where

$$\hat{V} = \frac{(\hat{k}_0 \times \hat{k}_{1T})}{|\hat{k}_0 \times \hat{k}_{1T}|} \times (-\hat{k}_0) \quad . \tag{C.1}$$

Then, the application of vector algebra leads to

$$\hat{n}_1 = \cos\alpha(-\hat{k}_0) + \sin\alpha\, \hat{V} \tag{C.2}$$

$$= \frac{1}{\sin(\alpha-\alpha')} \left[(\sin\alpha')\hat{k}_0 - (\sin\alpha)\hat{k}_{1T}\right] \quad . \tag{C.3}$$

By use of Snell's law, $\sin\alpha = n\sin\alpha'$, Eq. (C.3) simplifies to

$$n\hat{k}_{1T} = \hat{k}_0 - \{[(n^2-1)+\cos^2\alpha]^{\frac{1}{2}} - \cos\alpha\}\hat{n}_1 \quad . \tag{C.4}$$

We now apply the general vector relation, Eq. (C.4), to the coordinate system of Fig. 5. The components of $\hat{k}_0$ and $\hat{k}_{1T}$ are

$$\hat{k}_0 = -\sin\theta_0\, \hat{x} - \cos\theta_0\, \hat{z} \quad , \tag{C.5}$$

$$\hat{k}_{1T} = -(\sin\theta_{1T}\cos\phi_{1T}\,\hat{x} + \sin\theta_{1T}\sin\phi_{1T}\,\hat{y} + \cos\theta_{1T}\,\hat{z}) \quad , \tag{C.6}$$

where the minus sign in Eq. (C.6) arises because $\hat{k}_{1T}$ is defined relative to the -z direction (Fig. 2). In addition, we make use of the geometrical relation between the direction of the normal to a surface point and the slope components at that point:

$$\hat{n}_1 = \frac{(-s_{1x}\hat{x} - s_{1y}\hat{y} + \hat{z})}{(1+s_{1x}^2+s_{1y}^2)^{\frac{1}{2}}} \quad . \tag{C.7}$$

The substitution of Eqs. (C.5)-(C.7) into Eq. (C.4) yields expressions for (s_{1x}, s_{1y}) in terms of the incident and refracted angles:

$$s_{1x} = \frac{(\sin\theta_o - n\sin\theta_{1T}\cos\phi_{1T})}{(n\cos\theta_{1T} - \cos\theta_o)} \quad , \tag{C.8}$$

$$s_{1y} = -\frac{n\sin\theta_{1T}\sin\phi_{1T}}{(n\cos\theta_{1T} - \cos\theta_o)} \quad . \tag{C.9}$$

In Section V, it is necessary to transform integrations over slope variables into integrals over the angles (θ_{1T}, ϕ_{1T}). The corresponding limits of integration for the angle variables are not simply arrived at for the general rough surface. We see, from Eq. (C.4), that, for fixed $\hat{k}_o$ and n, the direction $\hat{k}_{1T}$ varies with the normal $\hat{n}_1$. Hence restrictions on the surface normal are equivalent to restrictions on $\hat{k}_{1T}$. The first restriction is $\hat{n}_1\cdot\hat{z} > 0$. From Eq. (C.4)-(C.6), we see that this condition is equivalent to

$$\frac{(n\cos\theta_{1T} - \cos\theta_o)}{\left\{[(n^2-1)+\cos^2\alpha]^{\frac{1}{2}} - \cos\alpha\right\}} > 0 \quad ,$$

or, because the denominator is always greater than zero,

$$n\cos\theta_{1T} > \cos\theta_o \quad . \tag{C.10}$$

The second condition is $(-\hat{k}_0 \cdot \hat{n}_1) > 0$. The substitution of Eqs. (C.5) and (C.7) into this condition yields

$$s_{1x} < \cot\theta_0 \quad . \tag{C.11}$$

We now substitute Eq. (C.8) into the left-hand side of the inequality. After algebra, the condition reduces to

$$\hat{k}_0 \cdot \hat{k}_{1T} > \frac{1}{n} \quad . \tag{C.12}$$

Since $n^{-1} = 3/4$ for water, this latter condition is a severe one.

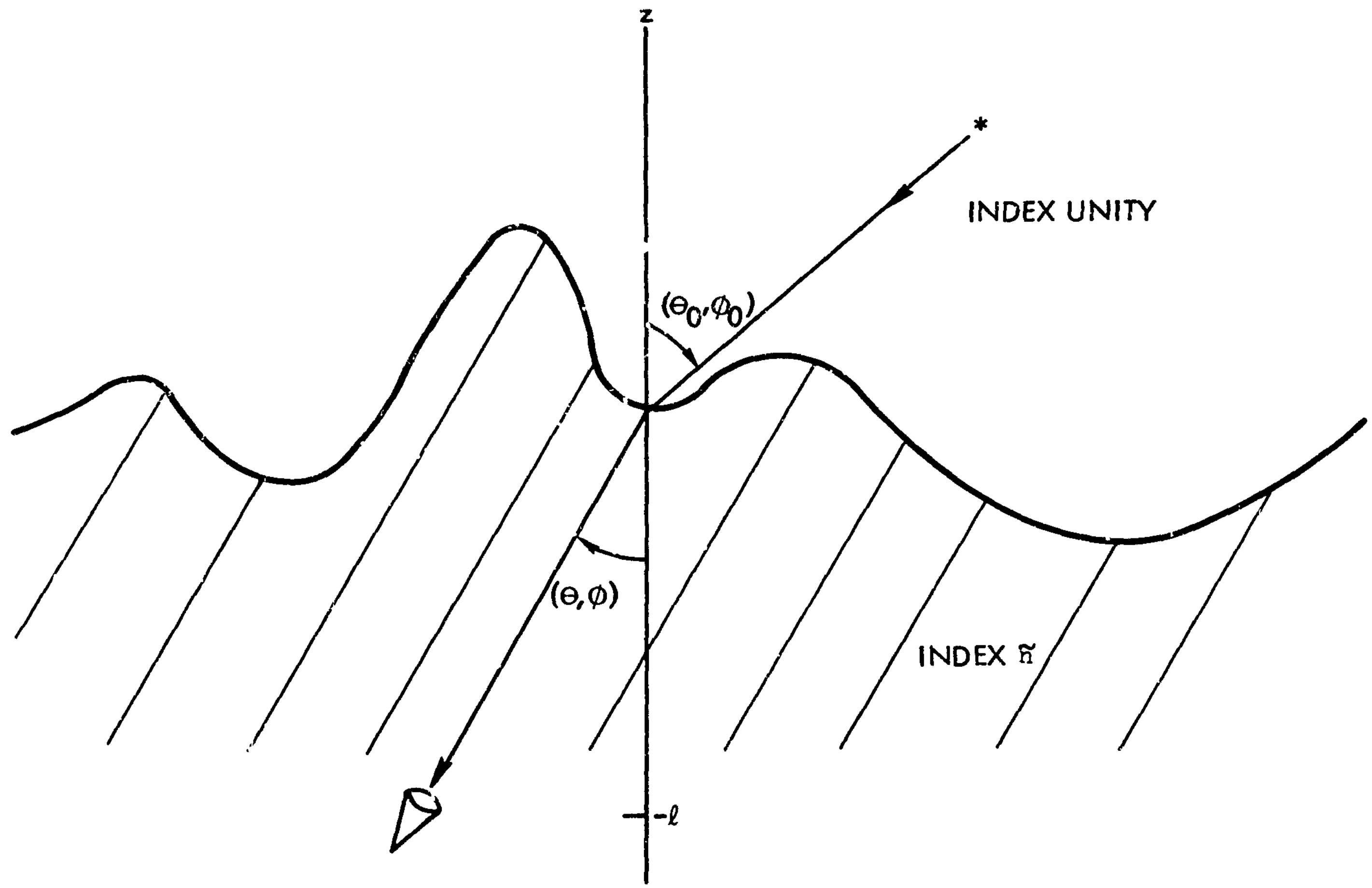

FIGURE 1
THE IDEALIZED TRANSMISSION CONFIGURATION

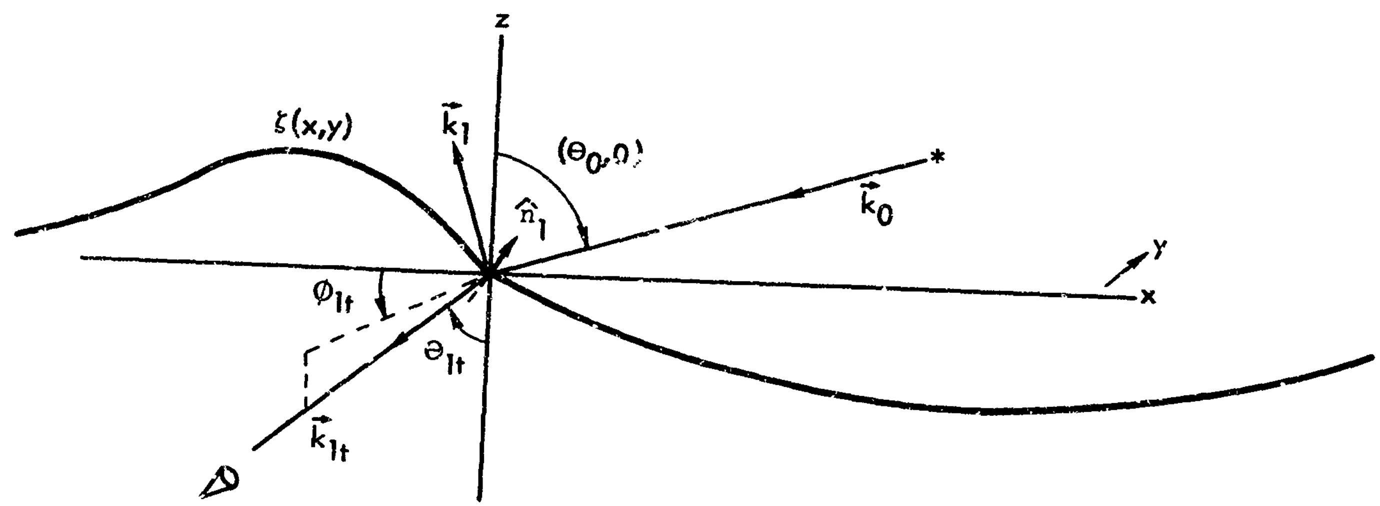

FIGURE 2
THE DIRECT TRANSMISSION PROCESS

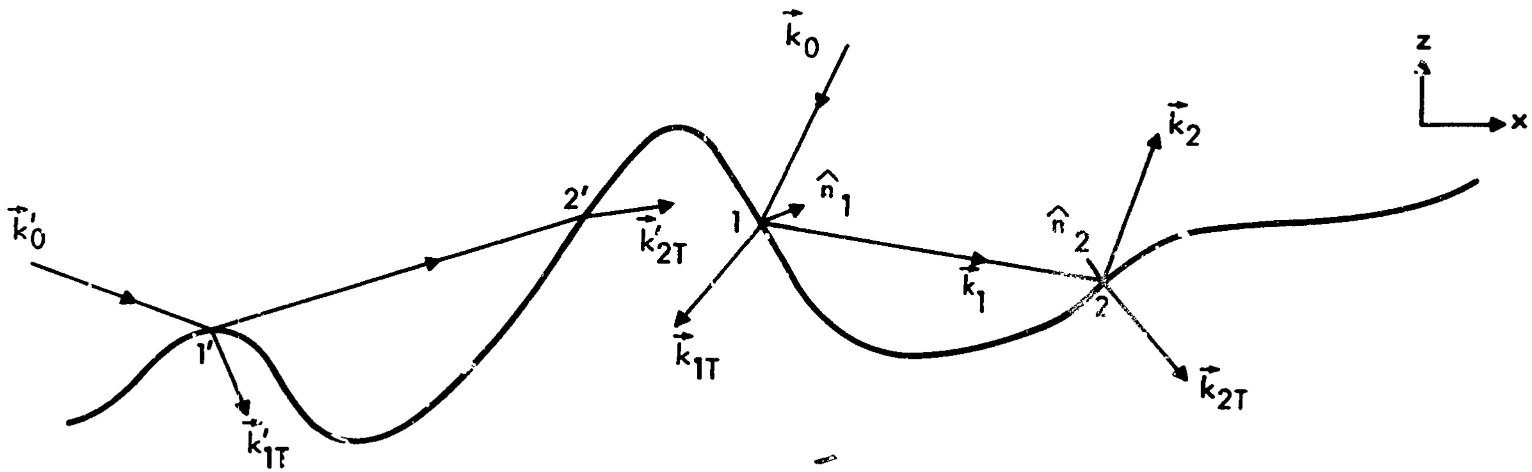

FIGURE 3

DOUBLE SCATTER FOLLOWED BY TRANSMISSION

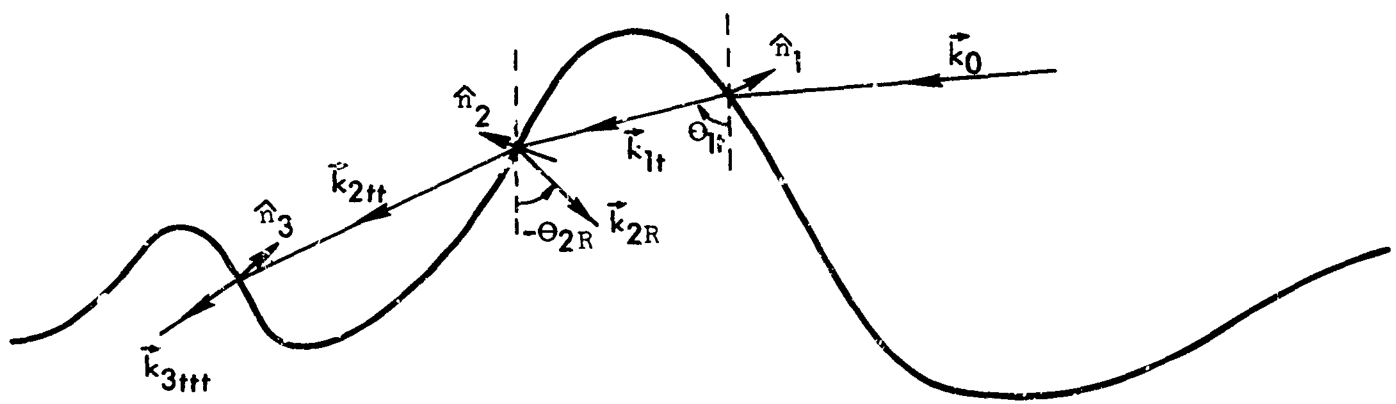

FIGURE 4

TRANSMISSION FOLLOWED BY INTERNAL REFLECTION.

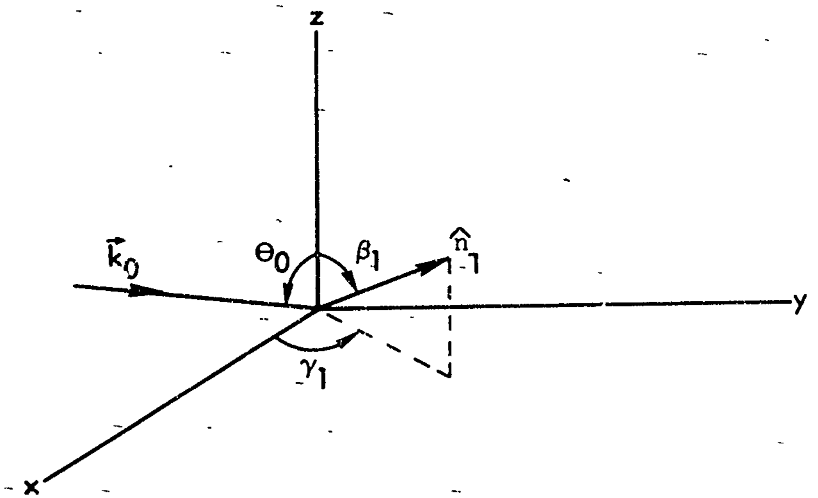

FIGURE 5
RADIATION INCIDENT ON AN ARBITRARILY ORIENTED PLANE SURFACE.

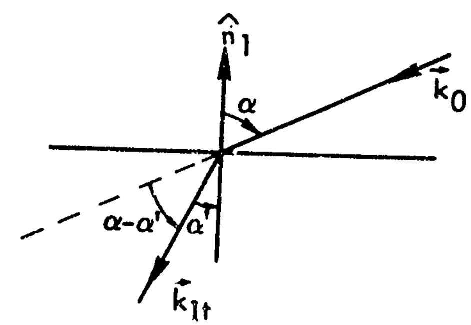

FIGURE 6
THE COPLANAR DIRECTIONS OF INCIDENCE, REFRACTION, AND LOCAL NORMAL.

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